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Endogenous Technological Change and Optimal Growth

PETER MURRELL*

1. Introduction

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It is useful to view optimal growth models s generalizations of the one-sector model see [6]). Uncertainty ([5]), a multi-sector echnology ([7] and [11]), and factor-ugmenting technological change ([4] and 10]) have all been added to the one-sector model. In each case, the models have a unique stationary state. In contrast, it will be shown here that an economy modelled using the Jorgenson-Griliches framework, in which technological change is represented by endogenous movements along the production function, has a multiplicity of stationary states.

*Department of Economics, University of Maryland, College Park, Md. 20742 U.S.A. This paper is a revised version of a chapter of my Ph.D. dissertation completed at the University of Pennsylvania. I would like to thank my advisor, Don Green, for valuable advice.

2. The Model

The model is a simple adaptation of the one-good model. The one change is that production potential increases because labor is employed to discover how to use new regions of the production function. A region is usable only if all regions with lower capital-labor ratios are usable. Thus, production of knowledge is cumulative and is represented by changes in γ, the highest capital-labor ratio available.[1] Movement in γ is given by: $\dot{\gamma} = G(L_2)$, where L_2 is labor used in the knowledge sector. Assume that $G'(L_2) > 0$, $G''(L_2) < 0$ and $G'(0) = B$ where $0 < B < \infty$. The finiteness of B implies that finite amounts of knowledge cannot be produced using infinitesimal amounts of labor.[2]

Society produces one tangible good which is consumed or invested. K is the size of the capital stock, c is per capita consumption, L_1 is labor used in the goods sector, and H is capital used in goods production. Use of capital is constrained by two factors. First, there is a limited amount available: $H \leq K$. Second,

[1] Most of the variables defined will be assumed to be functions of time, but the time argument will be omitted for the sake of brevity; thus, for example γ should be understood as $\gamma(t)$. A dot will be used to indicate a time derivative.

[2] The above assumptions are completely consistent with the usual assumptions on production functions made in the neoclassical literature.

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Most economists would agree with Kuznets' observation ([9], p. 286) that technological change is *the* distinguishing feature of modern economic growth. However, there is less agreement on the nature of that process. Abramovitz [1] views it as occurring through shifts in the production function. In contrast, Jorgenson and Griliches ([8], p. 272) claim that 96.7% of the changes in the pattern of productivity activity can be explained by movements along a given production function. Their treatment emphasizes the endogeneity of advances in knowledge and the costs of producing those advances ([8], p. 272).

Mirrlees ([10], p. 95) has observed that any applicable growth model must incorporate technological change. However, attempts to include technological change in optimal growth models have been highly stylized. The focus has been on shifts in production functions (see [4], [10]).

It is useful to view optimal growth models as generalizations of the one-sector model (see [6]). Uncertainty ([5]), a multi-sector technology ([7] and [11]), and factor-augmenting technological change ([4] and [10]) have all been added to the one-sector model. In each case, the models have a unique stationary state. In contrast, it will be shown here that an economy modelled using the Jorgenson-Griliches framework, in which technological change is represented by endogenous movements along the production function, has a multiplicity of stationary states.

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knowledge of how to use some capital-labor ratios is unavailable: $H/L_1 \leqq \gamma$.

The remaining assumptions are usual in the optimal growth literature. Output of goods is given by a neoclassical production function: $Y + Lc \leqq F(H, L_1)$, where Y is gross investment and L is population (= labor force). Investment is irreversible: $Y \geqq 0$. With δ signifying the depreciation rate, change in capital stock is $\dot{K} = Y - \delta K$. Finally, the use of labor is subject to its availability: $L_1 + L_2 \leqq L$.

The object is to maximize $\int_0^\infty Lu(c)e^{-\rho t}dt$, where ρ is the rate of pure time preference, and $u(c)$ is the instantaneous utility function with $u'(c) > 0$, $u''(c) < 0$, and $u'(0) = \infty$.

In order to anticipate potential criticism, I will comment on some inadequacies of the model and indicate whether or not these are important.

(i) Population is assumed constant. For a developed country, this may be reasonable and is certainly as plausible as the usual assumption of constant exponential growth.

(ii) Perhaps one should assume that knowledge production requires capital. In such a case, that capital would be qualitatively different from capital used to produce tangible goods. The model would then have three sectors, with attendant difficulties of analysis.

(iii) The model does not differentiate between labor in the two sectors. Such a differentiation would entail a description of how higher quality workers are produced from "raw" labor.

(iv) Capital used at higher capital-labor ratios might differ from that used at lower ratios. The present model's description tends to highlight this expectation. The non-recognition of variation in capital quality would be a criticism of all one-good models.

(v) It has been assumed that the production function is known over the entire range of capital-labor ratios, labor being used only to implement the processes. To incorporate discovery of the production function itself would take us far from an optimal growth framework.

3. The Necessary Conditions

To simplify the analysis, I will present the model in per capita terms. Define $y = Y/L$, $k = K/L$, $h = H/L$, $v = L_1/L$, $\beta = BL$, and $g(1 - v) = G([1 - v]L) = G(L_2)$. Then $g'(1-v) = G'([1 - v]L) \cdot L$ and thus $g'(0) = \beta$. As $F(\cdot)$ is neoclassical it exhibits constant returns to scale and one can define $f(H/L_1) = F(H/L_1, 1)$. Thus, previous equations become:

$$y + c \leqq vf\left(\frac{h}{v}\right) \tag{1}$$

$$\dot{k} = y - \delta k \tag{2}$$

$$h \leqq k \tag{3}$$

$$h/v \leqq \gamma \tag{4}$$

$$\gamma \leqq 1 \tag{5}$$

$$\dot{\gamma} = g(1 - v) \tag{6}$$

$$y \geqq 0 \tag{7}$$

The maximization criterion is then

$$\int_0^\infty u(c)e^{-\rho t}dt \tag{8}$$

To find necessary conditions formulate a Hamiltonian, $I(t)$, using auxiliary variables q_1 and q_2: $I(t)e^{\rho t} = u(c) + q_1(y - \delta k) + q_2 g(1 - v)$. Use of the Maximum Principle gives static (i) and dynamic (ii) conditions.

(i) For fixed k, γ, q_1 and q_2, maximize $I(t)$ subject to (1), (3), – (5), and (7). For this maximization, use the Kuhn-Tucker theorem by introducing auxiliary variables p, r_1, r_2, S_1, and S_2 and formulating:

$$J(t)e^{\rho t} = I(t)e^{\rho t} + p\left(vf\left(\frac{h}{v}\right) - y - c\right) + r_1(k - h) + r_2(\gamma v - h) + S_1 y + S_2(1 - v).$$

The static necessary conditions are then:

$$u'(c) = p \tag{9}$$

$$q_1 + S_1 = p \tag{10}$$

$$p\left[f\left(\frac{h}{v}\right) - \frac{h}{v} f'\left(\frac{h}{v}\right)\right] + r_2\gamma - S_2 - q_2 g'(1 - v) = 0 \tag{11}$$

$$pf'\left(\frac{h}{v}\right) - r_1 - r_2 = 0 \tag{12}$$

$$p\left(vf\left(\frac{h}{v}\right) - y - c\right) = 0, p \geqq 0 \tag{13}$$

$$r_1(k - h) = 0, r_1 \geqq 0 \tag{14}$$

$$r_2(\gamma v - h) = 0, r_2 \geqq 0 \tag{15}$$

$$S_1 y = 0, S_1 \geqq 0 \tag{16}$$

$$S_2(1 - v) = 0, S_2 \geqq 0 \tag{17}$$

(ii) The dynamic conditions are:

$$\frac{d}{dt}(q_1 e^{-\rho t}) = \frac{-\partial J(t)}{\partial k(t)},$$

and $\dfrac{d}{dt}(q_2 e^{-\rho t}) = \dfrac{-\partial J(t)}{\partial \gamma(t)}$. Therefore

$$\dot{q}_1 = (\rho + \delta)q_1 - r_1 \tag{18}$$

$$\dot{q}_2 = \rho q_2 - v r_2 \tag{19}$$

4. Stationary Points

At a stationary point, $\dot{\gamma} = 0$ and $\dot{k} = 0$. If $\dot{\gamma} = 0$, $v = 1$. If $\dot{k} = 0$, $y = \delta k$. Thus, from (9), (10), (13), and (15): $u'(c) = u'(f(k) - \delta k) = q_1 > 0$, and therefore $\dot{q}_1 = 0$.[3] Thus:

$$(\rho + \delta)q_1 = r_1. \tag{20}$$

From (12) and (18):

$$r_2 = q_1(f'(k) - \rho - \delta). \tag{21}$$

[3]One is only interested in optimal stationary points. Obviously, a stationary point with $k > h = \gamma$ will not be optimal. Therefore, one can take $k \geqq \gamma$.

$v = 1$ implies $S_2 \geqq 0$, therefore (11) is:

$$q_1(f(k) - kf'(k)) + r_2\gamma - q_2\beta \geqq 0.$$

From (21):

$$q_1(f(k) - kf'(k)) + \gamma q_1(f'(k) - \rho - \delta) - q_2\beta \geqq 0 \tag{22}$$

For those stationary points for which $\gamma = k$, (22) becomes

$$q_1(f(k) - k(\rho + \delta)) \geqq q_2\beta. \tag{23}$$

In order for (22) to hold at a stationary point $\dot{q}_2$ must be non-positive. Therefore $\rho q_2 \leqq r_2$, which from (21) implies:

$$\rho q_2 \leqq q_1(f'(k) - \rho - \delta). \tag{24}$$

Let us now find the optimal stationary points. Denote by k^* the point defined by $f'(k^*) = \rho + \delta$. Let $\gamma^* = k^*$. A stationary capital stock greater than k^* would imply, from (21), $r_2 < 0$ which contradicts (15) and is therefore non-optimal. In Result 1 (see appendix for all formal proofs), it is shown that a stationary point (k^0, γ^0) with $k^0 < \gamma^0$ and $k < k^*$ is not optimal. Define k^+ implicitly by: $\rho[f(k^+) - (\rho + \sigma)k^+] = \beta[f'(k^+) - \rho - \delta]$. Result 2 shows that a stationary point with $k < k^+$ is not optimal. k^+ has a further interpretation. If $q_2 \geqq 0$ in a stationary state, then $\dot{q}_2 = 0$, which implies, from (19), (21), and (23), that $k \geqq k^+$. To require $q_2 \geqq 0$ is reasonable since $q_2 < 0$ implies that a costless reduction in γ would improve welfare. Yet reducing γ can never be beneficial. Thus q_2 will henceforth be assumed to be non-negative.

Thus, $\{(k, \gamma)$: either $(\gamma = k)$ and $k^+ \leqq k \leqq k^*$, or $\gamma > k$ and $k = k^*\}$ is the set of optimal stationary points. One can show that optimal paths tend to a stationary state. As $\dot{\gamma} \geqq 0$, γ will either tend to a finite value or become infinite. If γ becomes infinite, either k becomes infinite or there is a time beyond which $\gamma > k$. A path with k infinite is non-optimal. If $\gamma > k$ after a time T, then paths with $\dot{\gamma}(t) > 0$ at some $t > T$ will not be optimal. Hence, no optimal path will have γ infinite. Thus, γ will tend to a finite value: let us call this value γ^0.

In the limit the production structure is exactly equivalent to that of the one-sector model. The production function becomes:

$$y + c \leqq f(k), \text{ for } k \leqq \gamma^0 \text{ and}$$

$$y + c \leqq f(\gamma^0), \text{ for } k > \gamma^0.$$

This is then a one-sector model, albeit with an unusual production function. Optimal paths in the one-sector model tend to a stationary state. Therefore, so will optimal paths in the present model.

5. The Movement to Stationary Points

The behavior of the economy over time and the optimal stationary state will depend on the initial state. In finding optimal paths, it is useful to demarcate areas of (k, γ) space and describe paths starting from each area. The description of paths starting from an area is immediately relevant to paths passing through that area. Diagram I shows the areas of (k, γ) space referred to in the ensuing analysis.

$\bar{k}$ is defined implicitly by: $\rho[f(\bar{k}) - \bar{k}f'(\bar{k})] = \beta[f'(\bar{k}) - \rho - \delta]$. Let $\bar{k} = \bar{\gamma}$. The significance of $\bar{k}$ will be evident in the ensuing analysis. The following sections correspond to the equivalently labelled area in Diagram I.

5.A $\{(\mathbf{k}, \gamma): \gamma \geqq \gamma^*, \mathbf{k} \geqq \mathbf{k}^*\}$

As $\dot{\gamma} \geqq 0$, paths from A go to stationary states with $\gamma \geqq \gamma^*$ and hence with $k = k^*$. First, examine movement from points where $\gamma > k$. The present model is equivalent to the one-good model with constraints added. Therefore, if the one-good optimal path of k is feasible in the present model, that path will also be optimal. In region A when $\gamma \geqq k$, the one-good optimal path of k is feasible. Therefore, an economy starting at (k^0, γ^0) with $k^0 > k^*$ and $\gamma^0 \geqq k^0$ will move towards (k^*, γ^0), but will not reach that point in finite time.

When $\gamma > k$, $\dot{q}_2 \geqq 0$ and $\dot{q}_2 = 0$ if and only if $q_2 = 0$. Therefore, when a path crosses the line $\gamma = k$, $q_2 = 0$. In the part of A where $k > \gamma$, $r_2 = pf'(h/v)$, $r_1 = 0$ and (11) becomes:

$$pf\left(\frac{h}{v}\right) - q_2 g'(1 - v) = S_2 \geqq 0. \quad (25)$$

When a path crosses $\gamma = k$, $S_2 > 0$ and $v = 1$.

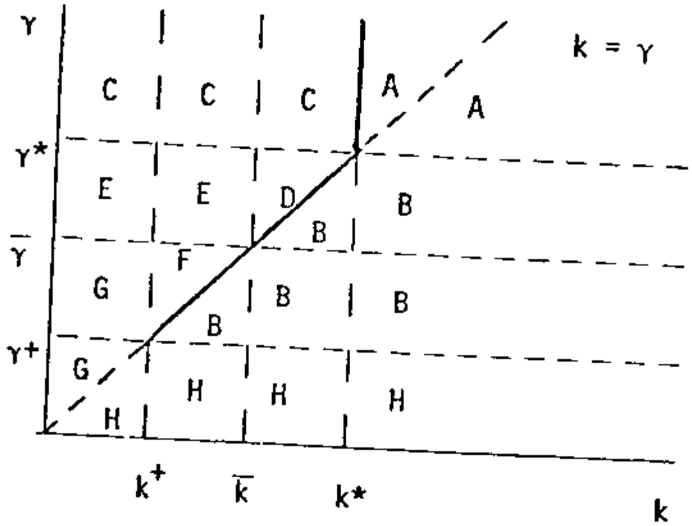

Diagram I

As (25) shows, earlier in time on such a path, S_2 would be smaller. Therefore at some earlier time, $S_2 = 0$, $v < 1$, and $\dot{\gamma} > 0$. Thus, paths in A with $k > \gamma$ could start with $\dot{\gamma} > 0$. However, $\dot{\gamma}$ becomes zero before $\gamma = k$. The paths described are depicted in Diagram II which appears at the end of Section 5.

5.B $\{(\mathbf{k}, \gamma): \mathbf{k} \geqq \gamma, \gamma^+ \leqq \gamma < \gamma^*\}$

Consider a path moving from B to the line $\gamma = k$. In B, $r_1 = 0$. Suppose $\dot{\gamma} > 0$ in a neighborhood of that line. Then (from (11) and (17)): $pf(\gamma) = q_2 g'(1 - v)$. As $p \geqq q_1$ and $\beta \geqq g'(\cdot)$, then $q_2 > q_1 f(\gamma)/\beta$. Therefore, as q_2, q_1 and γ are continuous, $q_2 \geqq q_1 f(\gamma)/\beta$ at the stationary point. Hence, at the stationary point (using (24)): $\beta(f'(\gamma) - \rho - \delta) \geqq \rho f(\gamma)$. Hence, $\gamma < \gamma^+$, which is a contradiction since $\gamma > \gamma^+$ in B. Thus, in B, $\dot{\gamma} = 0$ and $\dot{k} > 0$ in a neighborhood of the stationary state.

In Results 3 and 4, it is shown that on the path to the stationary state, $\dot{k}$ is negative and increasing. Using the same method as in Section 5.A, one can show that paths in B may have an early segment with $\dot{\gamma} > 0$. Hence, paths can pass from B to A. Paths in B will have a continuously declining capital stock until they reach the $\gamma = k$ axis. As $\dot{q}_1$ is bounded away from zero in B, the economy will reach the stationary point in finite time.

5.C $\{(\mathbf{k}, \gamma): \gamma \geqq \gamma^*, \mathbf{k} \leqq \mathbf{k}^*\}$

The path of k which is optimal in the one-good model is feasible in C. Therefore, that path is optimal. Thus, k rises monotonically to k^* but does not reach k^* in any finite time period.

5.D $\{(\mathbf{k}, \gamma): \mathbf{k} \leqq \gamma, \gamma < \gamma^*, \mathbf{k} \geqq \bar{\mathbf{k}}\}$

The significance of $\bar{k}$ can now be shown. A path in D moving towards a stationary state will have $\dot{k} > 0$ and $p = u'(C) = q_1$. Result 5 shows that in a neighborhood of the stationary state $\gamma = k/v = h/v$ if $\dot{\gamma} > 0$. Therefore, if $\dot{\gamma} > 0$:

$$q_1[f(\gamma) - \gamma f'(\gamma)] + r_2\gamma - q_2 g'(1 - v) = 0.$$

From the Maximum Principle, q_1, q_2, and γ are continuous. Result 5 shows that v is continuous. Thus, at the stationary state:

$$q_1[f(\gamma) - \gamma f'(\gamma)] \leqq q_2\beta. \qquad (26)$$

Combining (26) and (24):

$$\beta[f'(\gamma) - \rho - \delta] \geqq \rho[f(\gamma) - \gamma f'(\gamma)] \qquad (27)$$

which implies that $\gamma \leqq \bar{\gamma}$. Thus, the only stationary point in D which could be approached with $\dot{\gamma} > 0$ close to the stationary state is the point where $\gamma = \bar{\gamma}$.

Let us examine the movement to $(\bar{k}, \bar{\gamma})$. (26) will be a strict inequality if, and only if, $r_2 > 0$ as the path reaches $(\bar{k}, \bar{\gamma})$. Similarly (27) will be a strict inequality if $r_2 > 0$ when the path reaches $(\bar{k}, \bar{\gamma})$. Thus, if one can show that a path which has $\dot{\gamma} > 0$ close to $(\bar{k}, \bar{\gamma})$ and $r_2 = 0$ at the instant path reaches $(\bar{k}, \bar{\gamma})$ is impossible, then one can conclude that $\dot{\gamma} = 0$ within a neighborhood of $(\bar{k}, \bar{\gamma})$.

In regions D, E, or F when $\dot{\gamma} > 0$:

$$-r_2\gamma = q_1[f(\gamma) - \gamma f'(\gamma)] - q_2 g'(1 - v).$$

At the instant the path reaches $(\bar{k}, \bar{\gamma})$, $\dot{\gamma} = 0$. As v is continuous:

$$-\gamma\left\{\frac{dr_2}{dt}\right\} = \dot{q}_1[f(\gamma) - \gamma f'(\gamma)] - \dot{q}_2 g'(1 - v) + q_2 g''(1 - v)\dot{v}.$$

If $r_2 = 0$ at the instant the path arrives, then $\dot{q}_1 < 0$ and $\dot{q}_2 > 0$, and so $dr_2/dt > 0$. Thus $r_2 < 0$ before the path arrives, violating optimality conditions. Thus, (26) and (27) are strict inequalities. Therefore, every path in region D has $\dot{\gamma}$ 0 close to the stationary point.

Using the foregoing analysis, Result 6 shows that all optimal paths have $\dot{\gamma} = 0$ in D. Therefore, they also have $\dot{k} > 0$. In D, except at stationary points, $\dot{q}_1 = q_1[\rho + \delta - f'(k)]$. Hence, $\dot{q}_1$ is bounded away from zero at all times along all paths except the one along the line $\gamma = \gamma^*$. Thus, all stationary points except (k^*, γ^*) are reached in finite time.

5.E $\{(\mathbf{k}, \gamma): \bar{\gamma} \leqq \gamma < \gamma^*, \mathbf{k} < \bar{\mathbf{k}}\}$

From the previous discussion, we know paths leaving E have $\dot{\gamma} = 0$ and $\dot{k} > 0$, thereby moving into the region D. When paths enter D: $q_1[f(k) - kf'(k)] - q_2\beta = S_2 > 0$. Therefore, to find behavior in E, one must extend the above equation backwards in time. However, the specific behavior of that equation can only be found by specifying more concretely the nature of the production and utility functions. Result 7 shows that in some cases, paths will not move from F to E, but this is not a general result. Therefore, paths in E may contain an initial segment in which $\dot{\gamma} = 0$ and $\dot{k} > 0$, passing into D.

5.F $\{(\mathbf{k}, \gamma): \mathbf{k} \leqq \gamma, \gamma < \bar{\gamma}, \mathbf{k}^+ \geqq \mathbf{k}\}$

Result 8 shows that close to stationary point, $\dot{\gamma} > 0$ and $\dot{k} > 0$. Result 5 shows that $\gamma = k/v$.

In examining movement to a stationary point (k^0, γ^0), let r_2^0 be the stationary value of r_2 and r_2^1 be the value of r_2 at the instant the economy reaches (k^0, γ^0). As $\dot{q}_2 = 0$ at (k^0, γ^0): $r_2^0 = q_1[f'(k^0) - \rho - \delta] - \rho q_2$. Using the fact that v, γ, q_1, q_2, and k are continuous (see Result 5), the stationary state values obey:

$$q_1[f(k^0) - k^0 f'(k^0)] + r_2^1\gamma^0 - q_2\beta = 0.$$

Therefore:

$$(r_2^0 - r_2^1)\gamma^0 = q_1\left\{[f(k^0) - (\rho + \delta)k^0] - \frac{\beta}{\rho}[f'(k^0) - \rho - \delta]\right\} \geqq 0. \quad (28)$$

Thus, $(r_2^0 - r_2^1) > 0$ except when $(k^0, \gamma^0) = k^+, \gamma^+)$. r_2 must "jump" at all stationary points except the lowest one. In the instant before r_2 jumps: $\dot{q}_1 = (r_2^1 - r_2^0) = -\dot{q}_2$, where $(r_2^0 - r_2^1)$ is given in (28). Thus, $\dot{q}_1$ and $\dot{q}_2$ are bounded away from zero in the neighborhood of all stationary points in F except (k^+, γ^+). The economy reaches all F's stationary points, except (k^+, γ^+), in finite time.

Close to stationary points in F, $\gamma = k/v$. The set of necessary conditions can then be reduced to four differential equations; one for each of $\dot{k}$, $\dot{\gamma}$, $\dot{q}_1$, and $\dot{q}_2$. These equations are such that each rate of change is an explicit function of k, γ, q_1, and q_2. Paths going to a stationary point (k^0, γ^0) with $k^0 > k^+$ arrive in finite time. Thus, applying a well known theorem, one can say that an optimal path which satisfies the four equations and which goes to (k^0, γ^0) will exist and be unique.[4]

One cannot apply the same theorem when $(k^0, \gamma^0) = (k^+, \gamma^+)$ because paths reach this point only in the limit. Thus, except for (k^+, γ^+), each stationary point in Region F has one and only one path reaching it. On the final section of optimal paths in F, $\gamma = k/v$. Preceding that final section either $r_2 = 0$ (and $\gamma > k/v$) or $r_2 = pf'(h/v)$ (and $\gamma < k/v$). Thus, one must examine the behavior of r_2 to understand the economy's behavior.

Paths which become stationary near $(\bar{k}, \bar{\gamma})$ have r_2 close to zero at the instant they arrive at the stationary point. At that instant, as v and $\dot{k}$ are continuous, $\dot{\gamma} = 0$, $\dot{k}$ 0, and $\dot{v} = 0$. Differentiating (11), one finds, at that instant: $\dot{q}_1[f(k) - kf'(k)] + \dot{r}_2\gamma - \dot{q}_2\beta = 0$.

Therefore $\dot{r}_2 > 0$. With r_2 near zero and $\dot{r}_2 > 0$, a small distance backwards in time $r_2 = 0$ and hence $\gamma > k/v$. Stationary points with lower capital-labor ratios have paths leading to them with higher values of r_2/q_1 and lower values of $\dot{r}_2/q_1$ at the instant the paths become stationary (see Result 9). Hence, the lower the stationary capital-labor ratio, the lower is $\dot{r}_2/r_2$ at the instant the path becomes stationary. Thus, paths with higher stationary state values of k are more likely to have a phase in which $r_2 = 0$.

[4]See Boyce and DiPrima, p. 254. An assumption not stated in the text which is necessary for the theorem is that $u(\cdot)$, $f(\cdot)$, and $g(\cdot)$ should have continuous first and second derivatives.

One can make tentative predictions on the nature of optimal paths. Paths becoming stationary near $(\bar{k}, \bar{\gamma})$ have a phase with $\gamma = k/v$ preceded by one with $\gamma > k/v$. For paths with lower stationary values of k, the switch between these two phases occurs farther from the stationary point. One path in F will have $\gamma = k/v$ for all its length. Paths below this path will have the phase when $\gamma = k/v$ preceded by a phase with $\gamma < k/v$.

In Section 5.D, it is shown that there is a segment of line $\gamma = \bar{\gamma}$, beginning at $\bar{k}$ and ending at a lower value of k, which is not crossed or joined by any optimal path from F. Result 7 showed that this segment could be longer than $\{(k, \gamma): \gamma = \bar{\gamma}, k^{*} \leq k \leq \bar{k}\}$. Therefore, in some cases of the model, no optimal path will leave region F. However, this is not a general result and there may be paths which leave G passing through F on their way to E.

5.G $\{(\mathbf{k}, \gamma): \mathbf{k} < \gamma, \gamma < \bar{\gamma}, \mathbf{k} < \mathbf{k}'\}$

Paths in G are backward constructions of paths already discussed. Some paths in G go to E before becoming stationary in D. Others go through F to stationary points. Starting values of k and γ are not necessarily monotonically related to stationary values of these variables. (Note at this juncture, that Result 4 shows that no path leaves G, going through H to a stationary point in B).

5.H $\{(\mathbf{k}, \gamma)\ \gamma \leq \mathbf{k}, \gamma < \gamma^{+}\}$

Moving backwards from (k^{+}, γ^{+}), a path will lie along $\gamma = \gamma^{+}$ until the path drops down into H. This path will form a dividing line in H between paths which leave H by going into G and paths which go into B. Paths which begin underneath this dividing line move into G and thence into F or E. Paths which begin above the dividing line go to stationary points in B. Diagram II completes Section 5. This diagram summarizes Section 5 by showing a typical set of optimal paths as described in the foregoing analysis.

6. Discussion

The most striking feature of the results of Sections 4 and 5 is the multiplicity of stationary states. Multiple stationary states usually appear in optimal growth models which do not use the simple objective function (8), (see [12] and [13], for example). A single stationary point is the usual conclusion for one and two sector models which use objective functions of type (8). Hence it is important to ask: are the present results due to the introduction of endogenous technological change? In order to answer that question, the difference between technological knowledge and tangible goods must be identified.

Technological knowledge is produced with economic resources, as is tangible capital. Yet tangible capital depreciates because of aging and use. In contrast, knowledge does not decay. Hence, one must analyze the effect of the 'no decay' feature of technological knowledge. Let us therefore contrast the foregoing results with the results from a second model. The second model is identical to the first except that the former incorporates the (unrealistic) feature that knowledge decays. By comparing these two models, the significance of the distinguishing feature of technological knowledge, no decay, can be ascertained.

Introduce an exponential depreciation rate, θ, on knowledge. The one change in the model is that equation (6) becomes:

$$\dot{\gamma} = g(1 - v) - \theta\gamma. \qquad (29)$$

The only necessary condition which changes is (19):

$$\dot{q}_2 = (\rho + \theta)q_2 - vr_2. \qquad (30)$$

Knowledge must be continually produced in order to remain stationary. Therefore at a stationary point $v < 1$ and $\gamma = k/v$. Using the results of sector 3, let us analyze the characteristics of the stationary state. Using (29), at the stationary state: $\theta\gamma = g(1 - v)$. Since $g(\cdot)$ is monotonic and has an inverse function,

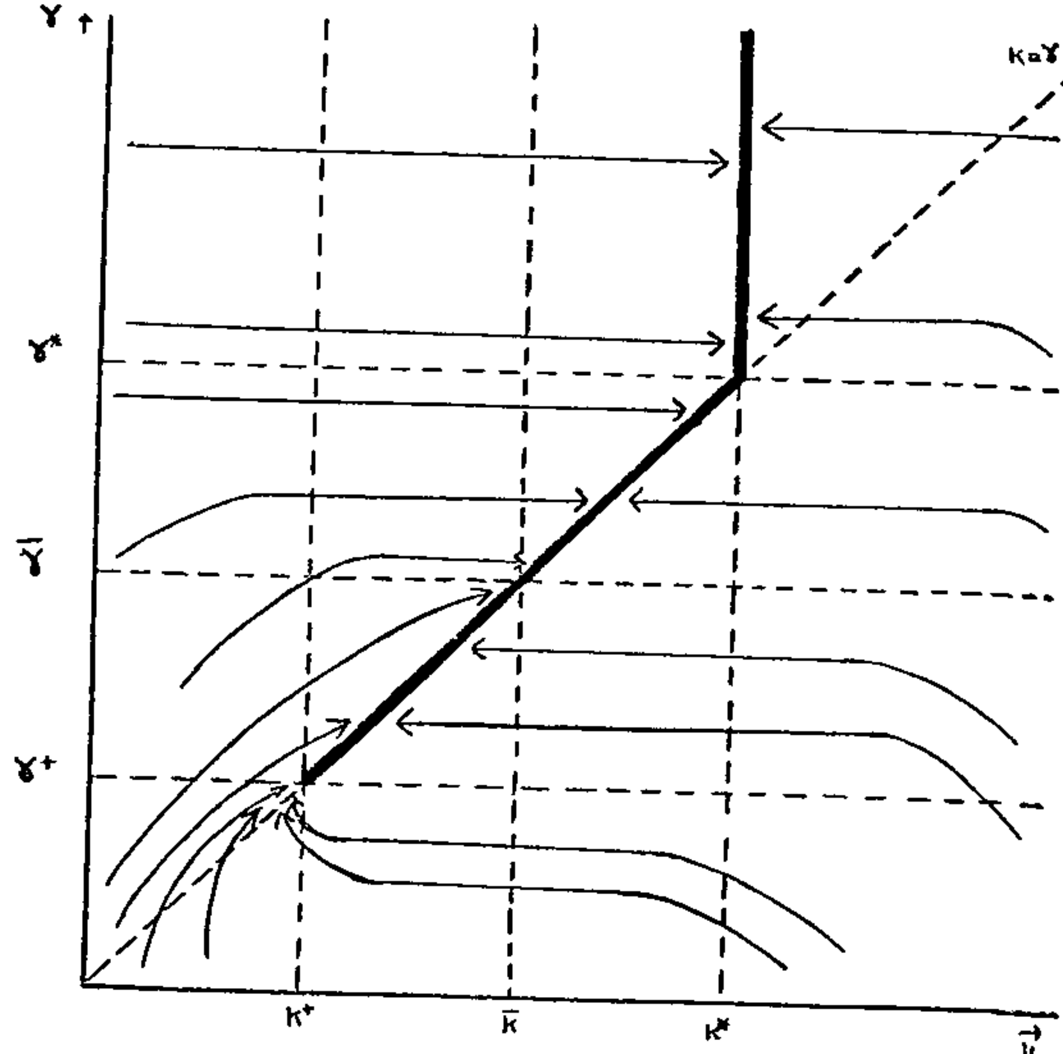

Diagram II

one can write solutions of $\dot{\gamma} = 0$ as $v = v(\gamma)$, where $v(0) = 1$ and $v'(\gamma) < 0$. Using the necessary conditions of section 3 with (29) and (30) replacing (6) and (19), one obtains at a stationary state:

$$q_1[f(\gamma) - \gamma f'(\gamma)] + q_1\gamma[f'(\gamma) - \rho - \delta] - \frac{v(\gamma)g'[1 - v(\gamma)]q_1}{(\rho + \theta)} \cdot [f'(\gamma) - \rho - \delta] = 0 \quad (31)$$

Let

$$M(\gamma) = f(\gamma) - (\rho + \delta) - \frac{v(\gamma)g'[1 - v(\gamma)]}{(\rho + \delta)}[f'(\gamma) - \rho - \delta].$$

Then, because $q_1 > 0$, equation (31) can be written as: $M(\gamma) = 0$. Using the fact that $v(0) = 1$ and $g'[1 - v(0)] = \beta$, one can show that $M(0) = -\infty$. Also, $M(\gamma^*) > 0$ and:

$$\frac{dM(\gamma)}{d\gamma} = M'(\gamma) = f'(\gamma) - \rho - \delta - \frac{[f'(\gamma) - \rho - \gamma]v'(\gamma)g'[1 - v(\gamma)]}{(\rho + \theta)} + \frac{v(\gamma)g''[1 - v(\gamma)]v'(\gamma)[f'(\gamma) - \rho - \delta]}{(\rho + \theta)} - \frac{v(\gamma)g'[1 - v(\gamma)]f''(\gamma)}{(\rho + \theta)}.$$

Thus, $M'(\gamma) > 0$ for $0 \leqq \gamma \leqq \gamma^*$. Therefore, there is a unique value of γ between 0 and γ^* which satisfies $M(\gamma) = 0$. This value is the stationary value of γ. Equation (29) shows that the stationary value of v, and hence k, will be unique.[5]

The model with decaying knowledge therefore produces very different results from the model with the more realistic assumption of non-decaying knowledge. The former model has a single stationary point, the latter a multiplicity. As the introduction of the non-decay feature is due to the introduction of technological change, one can expect that multiple stationary points will be a feature of optimal growth models which incorporate endogenous technological change of the Jorgenson-Griliches type.

7. Conclusion

In the present paper, a model of optimal economic growth incorporating endogenous technological change has been presented. The view of technological change used is that advanced by Jorgenson and Griliches [8] which is predicated on the endogeneity of research and development and represents changes in productive potential by movements along a production function. In this paper, a single parameter is used to represent knowledge accumulation. That parameter, which is raised by using labor, represents the highest level of sophistication of production processes available for use.

Analysis of the necessary conditions for optimality revealed that the model has a multiplicity of stationary states. Optimal paths were found for all possible starting points by the method of backward construction from stationary points. The initial point of a path determines both the stationary point and whether the path reaches that point in finite time. The set of stationary points reached in finite time is the set: $\{(k, \gamma): k = \gamma, k^+ \leqq k \leqq k^*\}$. The points (k^*, γ^*) and (k^+, γ^+) are reached in finite time by some paths but only in the limit by others.

It is interesting to ask to what particular phenomenon the distinctive characteristics of the present results can be attributed. Thus, in section six, one change in the model is made: the distinctive feature of technological change is removed. This removal results in a model which only has one stationary state. Thus, the multiplicity of stationary points is due to the introduction of endogenous technological change.

One feature of the present model not previously remarked upon is that stationary states are truly stationary: output per capita is constant. Other models have produced optimal paths on which productivity is always rising ([4] and [10]). However such paths are the result of the assumption that technological change is costless. When technological change requires resources, present consumption may be a more attractive alternative. In such a case, as described in sections 4 and 5, society chooses to stop producing knowledge.

The results of a simple optimal growth model cannot hope to give precise policy prescriptions to be applied immediately in a practical setting. One presents such models in the hope that they provide tentative, qualitative guides to policy. The results of this paper suggest that prescriptions from optimal

[5] The foregoing analysis makes the implicit assumption that $v(\gamma^*) \geq 0$.

growth models must be used with extreme caution. For one conclusion from the present model is that a small and perfectly reasonable change in assumptions may sharply change the qualitative nature of the results.

Appendix

Result 1. If the economy is in a stationary state (k^0, γ^0) at some time t^0, where $k^0 < \gamma^0$ and $k^0 < k^*$, then it will not be optimal to remain in this position.

Proof. Compare remaining at (k^0, γ^0) with moving to a stationary state $(k^0 + \Delta k, \gamma^0)$ where Δk is small and $k^0 + \Delta k < \gamma^0$. Let $c^0 = f(k^0) - \delta k^0$. Suppose that Δk of capital is accumulated at time t^0. The welfare cost of this accumulation is $\{u'(c^0)e^{-\rho t^0} \Delta k\}$. (One should note that the assumption that Δk is small allows one to ignore second-order effects. To be absolutely rigorous one should formulate the second-order effects and show that they become negligible in relation to the first order effects as Δk tends to zero. I have not deemed it necessary to pursue such rigor in the proofs in this Appendix.) The accumulation of Δk will allow extra stationary state production measuring $\{[f'(k^0) - \delta]\Delta k\}$. The value of the extra production

$$= \int_{t^0}^{\infty} e^{-\rho t} u'(c^0)(f'(k^0) - \delta)\Delta k dt$$

$$= u'(c^0)\frac{e^{-\rho t^0}}{\rho}(f'(k^0) - \delta)\Delta k$$

The move increases welfare if:

$$u'(c^0)\frac{e^{-\rho t^0}}{\rho}(f'(k^0) - \delta)\Delta k > u'(c^0)e^{-\rho t^0}\Delta k$$

that is, if $[f'(k^0) - \delta - \rho] > 0$. Hence, (k^0, γ^0) is not an optimal stationary state.

Result 2. Define k^+ by $\rho[f(k^+) - (\rho + \delta)k^+] = \beta[f'(k^+) - \rho - \delta]$. If the economy is stationary at (k^0, γ^0) at time t^0, where $\gamma^0 = k^0 < k^+$, then it is optimal to move.

Proof. Let $c^0 = f(k^0) = \delta k^0$. To move to a stationary state with higher k, both k and γ must be raised. Consider the following path to a higher stationary state:

(i) k drops instantaneously by Δk, therefore at this instant consumption increases by Δk.

(ii) Let $\gamma^0(1 - \Delta v) = k^0 - \Delta k$. Raise γ by reducing v by Δv for a time Δt.

(iii) Raise k by Δk plus the amount by which γ has risen, so that $\gamma = k$.

Now the gains and losses in welfare of moving along this path will be compared to staying at (k^0, γ^0). As the changes are all small, effects of higher order than $(\Delta v \Delta t)$ or $(\Delta k \Delta t)$ will be ignored. Along the path to the higher stationary state Δk of extra consumption occurs at time t^0 but Δk less occurs at time $t^0 + \Delta t$. Thus, on the path to the higher stationary state a gain occurs equal to the value of bringing forward k of consumption by Δt.

The gain due to the advance of Δk of benefits

$$= -\frac{d}{dt}(u'(c^0)\Delta k e^{-\rho t})\Delta t \Big|_{t=t^0}$$

$$= u'(c^0)e^{-\rho t^0}\gamma^0 \Delta v \Delta t \rho.$$

Now for a time period of length ΔT there is a loss of production, and thus consumption, for two reasons. First, labor is taken away from the tangible goods sector. Second, the level of capital stock is lower. The total loss, in welfare terms, due to these two effects is:

$$u'(c^0)e^{-\rho t^0}\Delta t\{\Delta v(f(k^0) - k^0 f'(k^0)) + \Delta k(f'(k^0) - \delta)\}$$

$$= u'(c^0)e^{-\rho t^0}\Delta t \Delta v\{f(k^0) - \delta k^0\}.$$

Along the path γ rises by $\beta \Delta v \Delta t$. Hence, there is a loss of consumption due to the fact that k also has to rise. The loss equals, in welfare terms, $u'(c^0)e^{-\rho t^0}\beta \Delta v \Delta T$.

There is a gain due to the rise in stationary state consumption equaling:

$$\int_{t^0}^{\infty} e^{-\rho t} u'(c^0)[f'(k^0) - \delta]\beta\Delta v \Delta t \, dt$$
$$= \frac{e^{-\rho t^0}}{\rho} u'(c^0)[f'(k^0) - \delta]\beta\Delta v \Delta t.$$

Adding together all gains and losses, the move leads to increased welfare if:

$$k^0\rho + (f'(k^0) - \delta)\frac{\beta}{\rho} > \beta + f(k^0) - \delta k^0$$

Hence, (k^0, γ^0) cannot be an optimal stationary point if $k^0 < k^+$.

Result 3. On paths to stationary points from B, k and hence c are continuous.

Proof. When $y > 0$, $q_1 = u'(c)$. Since q_1 is continuous c will also be continuous. If $v < 1$, then (11) and (12) combined show that v will be continuous (remember that γ, k, q_1, and q_2 are continuous). Thus, if $v \leq 1$ (1) and (2) show that $\dot{k}$ is continuous.

The result is less than trivial when one looks at the case when $y = 0$. Suppose that consumption jumps from c^0 to c^1 at a certain moment of time. At that moment, $q_1 \leq u'(c^0)$ and $q_1 \leq u'(c^1)$ because q_1 is continuous. When $c^0 < c^1$, then from (9) p falls. Hence, from (11) and (12) v will either fall or remain constant. Thus, from (1) and (13) y will fall. Hence, $u'(c^0) = q_1$, but as $c^0 < c^1$ then $u'(c^1) < u'(c^0)$, which implies that $u'(c^1) < q_1$, a contradiction. Similarly, when $c^0 > c^1$, then from (9), p rises. Hence, from (11) and (12) v will either rise or remain constant. Thus, from (1) and (13) y will rise. Hence, $u'(c^1) = q_1$, but as $c^0 > c^1$, $u'(c^0) < u'(c^1) = q_1$, which is a contradiction. Thus, consumption cannot jump. c is continuous, and from (11) and (12), v is continuous. Hence, $\dot{k}$ is continuous.

Result 4. For paths in Region B, $\dot{k} \leq 0$ always.

Proof. In discussing Region B, it has been shown that $\dot{k} \leq 0$ in the neighborhood of the stationary state. Suppose $\dot{k} > 0$ at some time and t^0 is the last instant of time at which $\dot{k} \geq 0$. Suppose t^1 is the time at which the stationary state is reached; obviously $t^1 > t^0$. In Region B, $\dot{q}_1 > 0$ always, and therefore $q_1(t^1) > q_1(t^0)$. Now $q_1(t^0) = u'(c(t^0))$ and $q_1(t^1) = u'(c(t^1))$, so $c(t^1) < c(t^0)$. In the stationary state $c(t^1) = f(\gamma(t^1)) - \delta k(t^1)$. At time t^0, $c(t^0) = v(t^0)f(\gamma(t^0)) - \delta k(t^0) - \dot{k}(t^0)$. Hence,

$$c(t^1) - c(t^0) = [f(\gamma(t^1)) - v(t^0)f(\gamma(t^0))] + \delta[k(t^0) - k(t^1)] + \dot{k}(t^0).$$

All three terms are non-negative. Hence, $c(t^1) \geq c(t^0)$: a contradiction. Thus, $\dot{k} \leq 0$ in B.

Result 5. A path near a stationary point in D, E, or F with $\dot{\gamma} > 0$ has $\gamma = k/v$.

Proof. To reach a stationary state from D, E, or F requires $\dot{k} > 0$ so that $q_1 = u'(c)$ and therefore c will be continuous. When $v < 1$ and $\gamma > k/v$, $r_2 = 0$, $r_1 = q_1 f'(k/v)$, thus

$$\dot{q}_2 = \rho q_2 > 0 \text{ and}$$

$$\dot{q}_1 = q_1\left[\rho + \delta - f'\left(\frac{k}{v}\right)\right] < 0.$$

Also from (11),

$$q_1\left(f\left(\frac{k}{v}\right) - \frac{k}{v}f'\left(\frac{k}{v}\right)\right) - q_2 g'(1-v) = S_2 = 0$$

at all points except, possibly, the stationary state. q_1, k, and q_2 being continuous, v is continuous in a neighborhood of the stationary point, except possibly for a 'jump' upwards at the stationary point. Now $c = vf(k/v) - \delta k - \dot{k}$, and in the neighborhood of the stationary point $v < 1$ and $\dot{k} > 0$. Therefore, if either v 'jumps' up to 1 or $\dot{k}$ 'jumps' down to 0, c will be discontinuous, which is not possible. Thus, both v and $\dot{k}$ will be continuous in a neighborhood of the stationary point, including at that point itself. Thus, at the stationary point (before S_2 jumps up at that point): $S_2 = q_1[f(k) - kf'(k)] - q_2\beta = 0$. In the neighborhood of the stationary point: