Econometric Analysis of Incomplete English Auction Models*

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Abstract

This paper studies identification and estimation of the distribution of bidder valuations in an incomplete model of English auctions. As in Haile and Tamer (2003) bidders are assumed to (i) bid no more than their valuations and (ii) never let an opponent win at a price they are willing to beat. Unlike the model studied by Haile and Tamer (2003), the requirement of independent private values is dropped, enabling the use of these restrictions on bidder behavior with affiliated private values, for example through the presence of auction specific unobservable heterogeneity. In addition, a semiparametric index restriction on the effect of auction-specific *observable* heterogeneity is incorporated, which, relative to nonparametric methods, can be helpful in alleviating the curse of dimensionality with a moderate or large number of covariates. The identification analysis employs results from Chesher and Rosen (2017) to characterize identified sets for bidder valuation distributions and functionals thereof.

1 Introduction

The path breaking paper Haile and Tamer (2003) (HT) develops bounds on the common distribution of valuations in an incomplete model of an open outcry English ascending auction in a symmetric independent private values (IPV) setting.

One innovation in the paper was the use of an incomplete model based on weak plausible restrictions on bidder behavior, namely that a bidder never bids more than her valuation and never

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allows an opponent to win at a price she is willing to beat. An advantage of an incomplete model is that it does not require specification of the mechanism relating bids to valuations. Results obtained using the incomplete model are robust to misspecification of such a mechanism. The incomplete model may be a better basis for empirical work than the button auction model of Milgrom and Weber (1982) sometimes used to approximate the process delivering bids in an English open outcry auction.

On the down side the incomplete model is partially, not point, identifying for the primitive of interest, namely the common conditional probability distribution of valuations given auction characteristics. HT derive bounds on this distribution and show how to use these bounds to perform inference on the distribution and interesting functionals such as the optimal reserve price. The question of the sharpness of those bounds was left open in HT.

The HT auction model was previously shown to fall in the class of Generalized Instrumental Variable (GIV) models introduced in Chesher and Rosen (2017), henceforth CR.[1] The results in that paper were applied to obtain a characterization of the identified set (sharp bounds) for the auction model as a leading example, and it was shown that there are observable implications additional to those given in HT that refine the bounds previously obtained.[2] The characterization of sharp bounds on valuation distributions was shown to comprise a dense system of infinitely many inequalities restricting not just the value of the distribution function via pointwise bounds on its level, but also restricting its shape as it passes between the pointwise bounds.

In this paper we expand the application of the intuitively appealing restrictions on bidder behavior invoked by HT to non-IPV settings. Theorems 1 and 2 in Section 3 provide general characterizations that do not require IPV. These results provide a framework for identification analysis incorporating further restrictions that appear in econometric models of auctions. Section 4 illustrates how the analysis can be applied to models that feature unobservable auction specific heterogeneity, a special and important class of models in which the IPV restriction does not hold.

Partial identification has been usefully applied to address other issues in auction models since HT. Tang (2011) and Armstrong (2013) both study first-price sealed bid auctions. Tang (2011) assumes equilibrium behavior but allows for a general affiliated values model that nests private and common value models. Without parametric distributional assumptions model primitives are generally partially identified, and bounds on seller revenue under counterfactual reserve prices and auction format are derived. Armstrong (2013) studies a model in which bidders play equilibrium strategies but have symmetric independent private values *conditional* on unobservable heterogeneity, and derives bounds on the mean of the bid and valuation distribution, and other interesting functionals. Aradillas-Lopez, Gandhi, and Quint (2013) study second price auctions that allow for

[1] The first analysis of the identifying power of the HT model using the GIV framework is in Chesher and Rosen (2015)

[2] In this paper we use the expression *identified set* to refer to *sharp bounds* throughout. Non-sharp bounds are referred to simply as bounds or *outer regions*.

correlated private values. Theorem 4 of Athey and Haile (2002) previously showed non-identification of the valuation distribution in such models, even if bidder behavior follows the button auction model equilibrium. Aradillas-Lopez, Gandhi, and Quint (2013) impose a slight relaxation of the button auction equilibrium, assuming that transaction prices are determined by the second highest bidder valuation. They combine restrictions on the joint distribution of the number of bidders and the valuation distribution with variation in the number of bidders to bound seller profit and bidder surplus.

The restrictions of the auction models we study are set out in Section 2. In Section 3 GIV models are introduced and the auction model is placed in the GIV context, and the identified set for such models is characterized. The identified set for an auction model with additive unobservable auction specific heterogeneity is characterized in Section 4. In Section 5 identification analysis is carried out in the presence of an index restriction on the effect of auction-specific *observable* heterogeneity. Relative to nonparametric methods, this can be helpful in alleviating the curse of dimensionality with a moderate or large number of covariates. Section 6 provides numerical illustrations of bounds on the effect of auction covariates on bidder valuations in such models. Section 7 concludes.

2 Model

We study open outcry English ascending auctions with a finite number of bidders, M, which may vary from auction to auction. The model imposes the slight simplifications that there is no reserve price and the minimum bid increment is zero. These conditions simplify the exposition and are easily relaxed.[3]

Auctions are characterized by a vector of observed final bids B, a vector of valuations V, and $Z = (X, M)$ comprising a vector of auction characteristics X and number of bidders M. B, V, Z are presumed to be realized on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with sigma algebra $\mathcal{A}$ endowed with the Borel sets on Ω. Valuations V are not observed. Final bids of those who place no bid are taken to be recorded at the lower bound of the support of bids, typically zero. Throughout the paper reference to "bids" should be taken to indicate final bids unless it is indicated otherwise. The way in which realizations of (B, Z) are observed across auctions renders their joint distribution identified.

The goal of our identification analysis is to determine what the joint distribution of (B, Z) reveals about the joint distribution of valuations conditional on Z. The notation $G_{V|Z}(\mathcal{S}|z)$ is used to denote the conditional probability that V is an element of $\mathcal{S}$, $\mathbb{P}[V \in \mathcal{S}|Z = z]$, while $\mathcal{G}_{V|Z}$ is used to denote the collection of such functions over all possible values of Z, $\mathcal{G}_{V|Z} \equiv \{G_{V|Z}(\mathcal{S}|z) : z \in \mathcal{R}_Z\}$, where $\mathcal{R}_Z$ denotes the support of Z. Inequalities involving random variables, such as those in Restrictions 1 and 2 below, and those stated in Lemma 1, are to be understood to mean these

[3]These conditions are also imposed in Appendix D of HT in which the sharpness of identified sets is discussed. As is the case in HT, with a reserve price r our analysis applies to the distribution of valuations truncated below at r.

inequalities hold $\mathbb{P}$ almost surely. For any random vector $W = (W_1, \dots, W_M)$, the notation $W_{m:M}$ denotes the m^{th} order statistic of W, so for example $W_{M:M} = \max(W)$ and $W_{1:M} = \min(W)$. The notation $\tilde{W}$ indicates the vector of ordered components of W, $\tilde{W} \equiv (W_{1:M}, \dots, W_{M:M})$.

Restriction 1. In an auction with M bidders, the final bids and valuations are realizations of random vectors $B = (B_1, \dots, B_M)$ and $V = (V_1, \dots, V_M)$ such that for all $m = 1, ..., M$, $B_m \leq V_m$ almost surely.

Restriction 2. In every auction the second highest valuation, $V_{M-1:M}$, is no larger than the highest final bid, $B_{M:M}$. That is, $V_{M-1:M} \leq B_{M:M}$ almost surely.

Restrictions 1 and 2 are the HT restrictions on bidder behavior. They admit the standard button auction equilibrium, but also allow for much more general bidder behavior, including departures from equilibrium. These restrictions place no requirements on the bidders' knowledge or beliefs about other bidders' valuations or strategies. Jump bids, often observed in ascending oral auctions – but ruled out in the standard button auction model – are allowed.

HT study the identifying power of Restrictions 1 and 2 when in addition bidder valuations are restricted to the independent private values paradigm, stated here as Restriction IPV.[4]

Restriction IPV (Independent Private Values)**.** There are independent private values conditional on auction characteristics $Z = z$ such that for all $z \in \mathcal{R}_Z$, the valuations of bidders are identically and independently continuously distributed conditional on $Z = z$.

The approach taken here applies identification analysis from CR, which automatically delivers sharp bounds for model primitives without need for a constructive proof of sharpness. Moreover, the analysis is applicable in the absence of the IPV restriction, and thereby establishes how the intuitively appealing restrictions 1-2 of HT can be much more broadly applied. We additionally consider the following exchangeability restrictions on unobserved valuations and observed bids, respectively.

Restriction EX-V (Exchangeability of Valuations)**.** Conditional on auction characteristics $Z = z$, bidder valuations $V = (V_1, ..., V_M)$ are exchangeable.

Restriction EX-B (Exchangeability of Bids)**.** Conditional on auction characteristics $Z = z$, observed final bids $B = (B_1, \dots, B_M)$ are exchangeable.

Restrictions EX-V and EX-B impose that valuations and bids, respectively, are exchangeable given auction characteristics z. Restriction EX-V places us within in the context of a model with symmetric bidders. It is less restrictive than Restriction IPV, in the sense that it is implied by Restriction IPV but does not imply it. Restriction EX-B imposes that bids are exchangeable, which is necessary to distinguish from Restriction EX-V in light of the incomplete model for bidder

[4] Restriction IPV as stated stipulates that valuations are continuously distributed, as is typically assumed in auction models. This is used in the ensuing derivations, but it is not essential for identification analysis and could be relaxed.

behavior mapping from valuations to bids. It is possible for example that Restriction EX-V holds so that valuations are symmetric, but that bids are not exchangeable due to heterogeneity in bidding strategies.

Exchangeability seems an appealing assumption when there is no observable information about specific bidder identities. It is an implication of restrictions that have been used in many applications, although it does rule out some interesting cases, such as when bidders have different observable types or when bidders form collusive bidding rings. Nonetheless, if bidder labels are randomly assigned in auction data, then exchangeability from the perspective of the econometrician is preserved. If bidder identities are observed, one might not want to impose these restrictions. In such cases the general approach to identification analysis put forward here remains applicable, albeit at the cost of added notation to distinguish bidder identities or types. In this paper we shall impose Restriction EX-V throughout, and impose Restriction EX-B in only some of our results. Restrictions 1-2 and IPV (and therefore EX-V) were imposed by HT. Restriction EX-B was not, and it is not required for their bounds to be valid.

3 Generalized Instrumental Variable models

This auction model falls in the class of Generalized Instrumental Variable (GIV) models introduced in Chesher and Rosen (2017). We use the results in CR to characterize the identified set (i.e. sharp bounds) for valuation distributions delivered by a joint distribution of M final bids.

A GIV model places restrictions on a process that generates values of observed endogenous variables, Y, given exogenous variables Z and U, where Z is observed and U is unobserved. The variables (Y, Z, U) take values on $\mathcal{R}_{YZU}$ which is a subset of a suitably dimensioned Euclidean space.

GIV models place restrictions on a structural function $h : \mathcal{R}_{YZU} \to \mathbb{R}$ which defines the admissible combinations of values of Y and U that can occur at each value z of Z which has support $\mathcal{R}_Z$. Admissible combinations of values of (Y, U) at $Z = z$ are zero level sets of this function, as follows.

$$\mathcal{L}(z; h) = \{(y, u) : h(y, z, u) = 0\}$$

For each value of U and Z we can define a Y-level set

$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\}$$

which is singleton for all u and z in complete models, but not in incomplete models such as those studied here. Likewise, for each value of Y and Z we can define a U-level set

$$\mathcal{U}(u, z; h) \equiv \{u : h(y, z, u) = 0\}. \tag{3.1}$$

GIV models place restrictions on such structural functions and also on a collection of conditional distributions

$$\mathcal{G}_{U|Z} \equiv \{G_{U|Z}(\cdot|z) : z \in \mathcal{R}_Z\}$$

whose elements are conditional distributions of U given $Z = z$ obtained as z varies across the support of Z, where $G_{U|Z}(\mathcal{S}|z)$ denotes the probability that $U \in \mathcal{S}$ conditional on $Z = z$.

3.1 Unordered Final Bids

In an auction of M bidders each bidder is characterized by an observed final bid B_m and private valuation V_m for the object at auction. The vector $(B_1, \ldots, B_M)$ denotes observed final bids for each of the M bidders in the auction, and the vector $V = (V_1, \ldots, V_M)$ denotes the corresponding unobserved private valuations. Neither vector B nor V need be ordered from smallest to largest or *vice-versa*. B_m and V_m correspond to the bid and valuation of the same bidder, but the order of the bids and valuation in B and V is otherwise arbitrary.

Valuations V_m are each restricted to have strictly increasing common marginal cumulative distribution function $F_z(\cdot)$ on their support conditional on auction characteristics $Z = z$. Valuations need not be independent. The support of each V_m is a subset of the extended real line.

For each $m = 1, ..., M$, define $U_m \equiv F_z(V_m)$, such that each variable U_m is marginally uniformly distributed on the unit interval. The vector $U \equiv (U_1, \ldots, U_M)$ plays the role of unobservable vector U for GIV analysis when considering unordered final bids. Statements predicated by $\forall m$ are to be understood to hold for all $m = 1, ..., M$.

The GIV level set of unobservable U corresponding to that in (3.1) given the HT assumptions from observed bids B is

$$\mathcal{U}(B, F_z) \equiv \left\{ u \in [0,1]^M : \forall m,\ F_z(B_m) \le u_m \wedge F_z\left(B_{m^*(B)}\right) \ge \max_{m \ne m^*(B)} u_m \right\}, \tag{3.2}$$

where $m^*(B)$ denotes the index of the winning bidder.

A GIV structural function which expresses these restrictions, with bid vector B taking the role of Y, is

$$h(B, z, U) = \sum_{m=1}^{M} \max((F_z(B_m) - U_m), 0) + \max(\max_{m \ne m^*(B)} U_m - F_z\left(B_{m^*(B)}\right), 0). \tag{3.3}$$

The auction model may thus be cast as a GIV model in which the structural function h is a known functional of the collection of conditional valuation distributions $\{F_z(\cdot) : z \in \mathcal{R}_Z\}$. We use the notation $\mathcal{F}$ to denote a collection of such conditional distribution functions, and F to denote those $\mathcal{F}$ permitted by the model, and which embody the researcher's prior information on the distribution functions $F_z(\cdot)$.[5]

The conditional distribution of U given $Z = z$, denoted G_z, is the joint distribution of M marginally uniform variates, i.e. a copula, and may vary with z. If however Restriction IPV is imposed, then the components of U are mutually independent conditional on Z and, for any set $\mathcal{S} \subseteq \mathcal{R}_U$, $G_z(\mathcal{S})$ is the probability that a random M-vector of independent uniform$(0,1)$ variates takes a value in the set $\mathcal{S}$. In the absence of restriction IPV, the *marginal* distribution of each component of U given Z is uniform$(0,1)$, but the components of U may be correlated. We use the notation $\mathcal{G}$ to denote a collection of copulas $\{G_z(\cdot) : z \in \mathcal{R}_Z\}$, and G to denote those collections $\mathcal{G}$ which are admitted by the model specification. For example, the conditional distributions G_z could each be left unrestricted across different values of z, their dependence on z could be parameterized through an index function, or they could be explicitly parameterized by an M-dimensional copula.

Applying Theorem 2 and Lemma 1 of CR, the identified set for the pair $F_z(\cdot)$ and $G_z(\cdot)$ are those such that for all sets $\mathcal{S}$ in a collection of test sets $\mathsf{Q}(h, z)$ the following inequality is satisfied almost surely

$$G_z(\mathcal{S}) \geq \mathbb{P}\left[\mathcal{U}(Y, Z; h) \subseteq \mathcal{S} | Z = z\right]. \tag{3.4}$$

The collection of test sets $\mathsf{Q}(h, z)$ is defined in Theorem 3 of CR. It comprises certain unions of the members of the collection of U-level sets $\mathcal{U}(y, z; h)$ obtained as y takes values in the conditional support of Y given $Z = z$.[6] The following theorem provides the formal result for the auction model where $\mathcal{U}(B, F_z)$ takes the place of $\mathcal{U}(Y, Z; h)$.

Theorem 1 *Let $\mathcal{F} \in \mathsf{F}$, $\mathcal{G} \in \mathsf{G}$, and Restrictions 1-2 and EX-V hold. The identified set for*

$$\{(F_z(\cdot), G_z(\cdot)) : z \in \mathcal{R}_Z\}$$

are those collections of conditional distributions admitted by F and G such that for almost every $z \in \mathcal{R}_Z$, for all $\mathcal{S}$ that are unions of sets of the form $\mathcal{U}(b, F_z)$, $b \in \mathcal{R}_B$:

$$\mathbb{P}\left[\mathcal{U}(B, F_z) \subseteq \mathcal{S} | Z = z\right] \leq G_z(\mathcal{S}). \tag{3.5}$$

Theorem 1 directly applies results from CR to characterize the identified set for the marginal distribution of valuations $F_z(\cdot)$ and their copula $G_z(\cdot)$ across values of z. It imposes Restrictions 1 and 2 on bidder behavior and Restriction EX-V, without recourse to independence of bidder valuations or exchangeability of observed bids. A feature of the Theorem is thus its generality.

Auction models employed in empirical work typically impose additional restrictions, which will

[5] For example, a model could restrict each $F_z(\cdot)$, $z \in \mathcal{R}_Z$ to a parametric family, or it could restrict $F_z(\cdot)$ to be invariant with respect to certain components of z.

[6] In general the collection $\mathsf{Q}(h, z)$ contains all sets that can be constructed as unions of sets, (3.1), on the support of the random set $\mathcal{U}(Y, Z; h)$. In particular models some unions can be neglected because the inequalities they deliver are satisfied if inequalities associated with other unions are satisfied. There is more detail and discussion in CR.

further refine the set given by (3.5), and which, thinking of implementation, may facilitate estimation and inference. The remainder of the paper studies models with additional restrictions, and develops characterizations of the resulting identified sets that refine those obtained using Theorem 1. To do this we turn to bounds on valuation distributions derived from knowledge of only the distribution of *ordered* final bids, that is using order statistics of the bid distribution, as were also employed in HT.

We show in the next section that when bids are exchangeable so that Restriction EX-B holds, then there is no loss of information in having knowledge of only the distribution of ordered final bids, in the sense that the identified set obtained using the distribution of ordered final bids is sharp and is the same as that obtained using the distribution of unordered final bids. If Restriction EX-B does not hold, then the bounds derived from information in the distribution of ordered bids still apply, but they may not be sharp relative to the bounds obtained from the distribution of unordered final bids.

3.2 Ordered Final Bids

Let $Y = (Y_1, ..., Y_M)$ denote ordered final bids, so that

$$Y_m \equiv B_{m:M}, \qquad m \in \{1, \ldots, M\}.$$

It is convenient to write the ordered valuations as functions of uniform order statistics. Define

$$\tilde{V} \equiv (V_{1:M}, \ldots, V_{M:M}), \quad \tilde{U} \equiv (U_{1:M}, \ldots, U_{M:M}) = (F_z(\tilde{V}_1), \ldots, F_z(\tilde{V}_M)),$$

to be ordered versions of V and U, respectively. The components of $\tilde{V}$ and $\tilde{U}$ order those of V and U from smallest to largest, so the m^{th} component of each, $\tilde{V}_m$ and $\tilde{U}_m$, respectively, are the m^{th} order statistics of V and U. The distribution of $\tilde{U}$ is therefore that of the order statistics of M uniform$(0, 1)$ but possibly dependent random variables. Without placing restrictions on the joint distribution of valuations, many such distributions of $\tilde{U}$ are possible, depending on the copula of V, $G_z(\cdot)$. The notation $\widetilde{G}_z(\mathcal{S})$ is used to denote the probability placed on the event $\tilde{U} \in \mathcal{S}$ when this copula is $G_z(\cdot)$. The admissible collection of joint distributions for U conveyed by $G_z(\cdot)$ restricts the collection of admissible $\widetilde{G}_z(\cdot)$.[7]

The following inequalities involving the order statistics of final bids B and latent valuations V set out in Lemma 1 are a consequence of Restrictions 1 and 2. The proof of the lemma, like all other proofs, is provided in Appendix A.

[7]Thus the notation $G_z(\mathcal{S})$ and $\widetilde{G}_z(\mathcal{S})$ distinguises between a joint distribution of M marginally uniform$(0, 1)$ random variables, and the joint distribution of the *order statistics* of M marginally uniform$(0, 1)$ random variables, respectively. The dependence structure among the M marginally uniform$(0, 1)$ components need not be known.

Lemma 1 *Let Restrictions 1-2 hold. Then for all m and M*

$$B_{m:M} \leq V_{m:M} \tag{3.6}$$

$$B_{M:M} \geq V_{M-1:M}. \tag{3.7}$$

In similar manner to HT, we can base identification analysis on the restrictions (3.6) and (3.7) on bid and valuation order statistics. We show that in fact when final bids are exchangeable, application of the GIV analysis to *ordered* bids and valuations delivers the same sharp bounds as are obtained using the information in the distribution of unordered final bids. The result appears in Theorem 2.

The restrictions (3.6) and (3.7) of Lemma 1 can be written as

$$\forall m, \quad Y_m \leq \tilde{V}_m = F_z^{-1}(\tilde{U}_m) \text{ and } Y_M \geq \tilde{V}_{M-1} = F_z^{-1}(\tilde{U}_{M-1})$$

and, on applying the increasing function $F_z(\cdot)$, they are as follows:

$$\forall m, \quad F_z(Y_m) \leq \tilde{U}_m \text{ and } F_z(Y_M) \geq \tilde{U}_{M-1}. \tag{3.8}$$

A GIV structural function which expresses these restrictions is

$$h(Y, z, \tilde{U}) = \sum_{m=1}^{M} \max((F_z(Y_m) - \tilde{U}_m), 0) + \max((\tilde{U}_{M-1} - F_z(Y_M)), 0). \tag{3.9}$$

The structural function h is known up to the collection of distributions $\mathcal{F} \equiv \{F_z(\cdot) : z \in R_Z\}$, so we replace h in the definition of random sets $\widetilde{\mathcal{U}}(Y, z; h)$, instead expressing $\tilde{U}$-level sets as:

$$\widetilde{\mathcal{U}}(Y, F_z) = \left\{ \tilde{u} : \left(\bigwedge_{m=1}^{M} (\tilde{u}_m \geq F_z(Y_m)) \right) \wedge (F_z(Y_M) \geq \tilde{u}_{M-1}) \right\}, \tag{3.10}$$

it being understood that for all m, $\tilde{u}_m \geq \tilde{u}_{m-1}$.[8] These are not singleton sets.

Figure 2 illustrates for the 2 bidder case. The $\tilde{U}$-level set $\widetilde{\mathcal{U}}((y_1, y_2), F_z)$ is the shaded rectangle below the 45° line. In Figure 2 all of, and only, the values $(\tilde{u}_1, \tilde{u}_2)$ in the shaded rectangle are capable of delivering ordered final bids such that $F_z(y_1) = 0.2$ and $F_z(y_2) = 0.6$.

With the auction model cast as a GIV model involving ordered bids Y and latent variables $\tilde{U}$ such that $\tilde{U}$ and Z are independently distributed, application of Theorem 4 of CR gives the following result.

[8] If there was a minimum bid increment Δ, then (3.10) would have $F_z(y_M + \Delta)$ in place of $F_z(y_M)$.

Theorem 2 *Let $\mathcal{F} \in \mathsf{F}$, $\mathcal{G} \in \mathsf{G}$, and Restrictions 1-2, EX-V, and EX-B hold. The identified set for $(F_z(\cdot), G_z(\cdot) : z \in \mathcal{R}_Z)$ are those collections of conditional distributions admitted by* F *and* G *such that for almost every $z \in \mathcal{R}_Z$, for all $\mathcal{S}$ that are unions of sets of the form $\widetilde{\mathcal{U}}(y, F_z)$ for $y \in \mathcal{R}_Y$.*

$$\mathbb{P}\left[\widetilde{\mathcal{U}}(Y, F_z) \subseteq \mathcal{S}|z\right] \leq \widetilde{G}_z(\mathcal{S}), \tag{3.11}$$

where $\widetilde{G}_z(\mathcal{S})$ denotes the probability that $\tilde{U} \in \mathcal{S}$ conditional on $Z = z$.

Relative to Theorem 1, Theorem 2 simplifies characterization of the identified set when Restriction 3 holds. The simplification lies in that the collection of inequalities defining the identified set involves only probabilities featuring *ordered* bids. When considering test sets, one need only consider unions of sets (3.10) in which $\tilde{u}_1 \leq \cdots \leq \tilde{u}_M$. This is an $M!$ fold reduction in the number of sets whose unions must be considered when constructing test sets $\mathcal{S}$.

In Chesher and Rosen (2017) the identified set for $F_z(\cdot)$, each $z \in \mathcal{R}_Z$, was characterized when in addition Restriction IPV holds. With this restriction in place $G_z(\cdot)$ is known and $\widetilde{G}_z(\mathcal{S})$ is the probability that the order statistics of M i.i.d. uniform random variables belong to the set $\mathcal{S}$, which is easily computed. In that paper a numerical illustration was provided for a two-bidder auction model in which the sharp characterization of the identified set for $F_z(\cdot)$ refined the bounds previously available.

4 Auction Specific Unobservable Heterogeneity

In this section we consider a model in which auction specific unobserved heterogeneity affects bidders' valuations. Allowing for such heterogeneity is important when the good being auctioned has some features observed by bidders, but not observed by the researcher, that have a common effect on each bidders' value. The unobserved variable could for example be a measure of quality unavailable to the researcher.

Unobservable auction-specific heterogeneity has featured in a variety of auction models in the recent literature, with examples including Krasnokutskaya (2011), Armstrong (2013), Roberts (2013), and Quint (2015), where bidders exhibit behavior consistent with some definition of equilibrium. This section examines the use of Restrictions 1 and 2, originally used by HT studying open outcry ascending auctions under the IPV restriction, to study identification when auction specific unobservable heterogeneity is allowed.

Bidder valuations are now restricted to comprise the sum of a private value component V_m^I and an auction-specific component V^+ as follows.[9]

[9] Valuations may measured on a logarithmic or other scale defined on the extended real line through application of any strictly monotone transformation as is done in Section 6.

Restriction UAH (Unobservable Auction Heterogeneity). Valuations in an M bidder auction are given by $V = (V_1, ..., V_M)$ where for each m, $V_m = V_m^I + V^+$, where $V_m^I \sim F_z(\cdot)$ and $V^+ \sim F_{z+}(\cdot)$. Both $F_z(\cdot)$ and $F_{z+}(\cdot)$ are strictly increasing on their supports.

Restriction UAH introduces auction-specific unobservable heterogeneity. Both distributions $F_z(\cdot)$ and $F_{z+}(\cdot)$ may vary with z. The joint distribution of $V^I \equiv \left(V_1^I, \ldots, V_M^I\right)$ and V^+ is left unrestricted. The notation $\mathcal{F}_+$ is used to denote a particular collection $\{F_{z+}(\cdot) : z \in \mathcal{R}_Z\}$ and F_+ denotes those collections of $\mathcal{F}_+$ admitted by the model specification.[10] The restriction that $F_{z+}(\cdot)$ be strictly increasing can be easily relaxed at the cost of some notational modification, but without substantive change to the subsequent results.[11]

Define the random $M+1$ vectors U and $\tilde{U}$ as

$$U \equiv \left(F_z\left(V_1^I\right), ..., F_z\left(V_M^I\right), F_{z+}\left(V^+\right)\right), \quad \tilde{U} \equiv \left(F_z\left(V_{1:M}^I\right), ..., F_z\left(V_{M:M}^I\right), F_{z+}\left(V^+\right)\right).$$

such that the first M components of $\tilde{U}$ have a joint distribution which is the joint distribution of the order statistics of M (possibly dependent) marginally uniform$(0,1)$ variables. The $(M+1)^{th}$ component of both U and $\tilde{U}$ is marginally uniform$(0,1)$.

As in Section 3.2 we continue to work with bid order statistics, with each m^{th} bid order statistic denoted $Y_m = B_{m:M}$. The notation $G_z(\mathcal{S})$ is used to denote the probability that $U \in \mathcal{S}$ conditional on $Z = z$, with $\widetilde{G}_z(\mathcal{S})$ denoting the probability that $\tilde{U} \in \mathcal{S}$ conditional on $Z = z$ for any $\mathcal{S} \in [0,1]^{M+1}$. $\mathcal{G}$ denotes a particular collection $\{G_z(\cdot) : z \in \mathcal{R}_Z\}$ and G denotes such collections which are admitted by the model specification. $G_z^-(\mathcal{S})$ is used to denote the probability that the first M components of $\tilde{U}$ belong to any set $\mathcal{S} \subseteq [0,1]^M$.

The set of feasible values of unobservable $\tilde{U}$ as a function of Y is given by

$$\widetilde{\mathcal{U}}(Y; F_z, F_{z+}) \equiv \left\{ \tilde{u} \in \mathcal{R}_{\tilde{U}} : \begin{array}{c} \forall m,\, Y_m \leq F_z^{-1}(\tilde{u}_m) + F_{z+}^{-1}(\tilde{u}_{M+1}), \\ \wedge \quad Y_M \geq F_z^{-1}(\tilde{u}_{M-1}) + F_{z+}^{-1}(\tilde{u}_{M+1}) \end{array} \right\}, \tag{4.1}$$

where $\mathcal{R}_{\tilde{U}}$ denotes the collection of vectors $\tilde{u} \in [0,1]^{M+1}$ on which $\tilde{u}_1 \leq \cdots \leq \tilde{u}_M$. Equivalently

$$\widetilde{\mathcal{U}}(Y; F_z, F_{z+}) \equiv \left\{ \tilde{u} \in \mathcal{R}_{\tilde{U}} : \max_{m=1,\ldots,M} Y_m - F_z^{-1}(\tilde{u}_m) \leq F_{z+}^{-1}(\tilde{u}_{M+1}) \leq Y_M - F_z^{-1}(\tilde{u}_{M-1}) \right\}.$$

With a slight abuse of notation, for any set $\mathcal{Y} \subseteq \mathcal{R}_Y$ let

$$\widetilde{\mathcal{U}}(\mathcal{Y}; F_z, F_+) \equiv \bigcup_{y \in \mathcal{Y}} \widetilde{\mathcal{U}}(y; F_z, F_{z+}), \tag{4.2}$$

[10] For example, F_+ could specify that $F_{z+}(\cdot)$ does not vary with z.

[11] For example, the analysis could then be carried out subject to minor modification with the final component of both U and $\tilde{U}$ simply defined as V^+, and $F_{z+}^{-1}(\tilde{u}_{M+1})$ in the inequalities below replaced with $\tilde{u}_{M+1}$.

denote the union of level sets $\widetilde{\mathcal{U}}(y;F_z,F_{z+})$ across $y\in\mathcal{Y}\subseteq\mathcal{R}_Y$.

From Theorem 2 it follows that under Restrictions 1-2 and UAH, for each z the identified set for $(F_z(\cdot),F_{z+}(\cdot),G_z(\cdot))$ are those that satisfy

$$\forall\mathcal{Y}\subseteq\mathcal{R}_Y,\quad \mathbb{P}[Y\in\mathcal{Y}|z]\leq\widetilde{G}_z\left(\widetilde{\mathcal{U}}(\mathcal{Y};F_z,F_{z+})\right).$$

For any $\mathcal{Y}\subseteq\mathcal{R}_Y$ the probability $\mathbb{P}[Y\in\mathcal{Y}|z]$ is identified from knowledge of the joint distribution of (Y,Z). The probability $\widetilde{G}_z\left(\widetilde{\mathcal{U}}(\mathcal{Y};F_z,F_{z+})\right)$ is the conditional probability of the event that $\tilde{U}\in\widetilde{\mathcal{U}}(\mathcal{Y};F_z,F_{z+})$ for some $y\in\mathcal{Y}$. That is

$$\tilde{U}\in\widetilde{\mathcal{U}}(\mathcal{Y};F_z,F_{z+})\Leftrightarrow\exists y\in\mathcal{Y}\text{ s.t. }\tilde{U}\in\widetilde{\mathcal{U}}(y;F_z,F_{z+}),$$

and consequently,

$$\widetilde{G}_z\left(\widetilde{\mathcal{U}}(\mathcal{Y};F_z,F_{z+})\right)=\widetilde{G}_z(\mathcal{E}(\mathcal{Y})),$$

where

$$\mathcal{E}(\mathcal{Y})\equiv\left\{\tilde{u}\in\mathcal{R}_{\tilde{U}}:\min_{y\in\mathcal{Y}}\min\left\{\begin{array}{c}\max_{m=1,\ldots,M}y_m-F_z^{-1}(\tilde{u}_m)-F_{z+}^{-1}(\tilde{u}_{M+1}),\\ F_{z+}^{-1}(\tilde{u}_{M+1})-\left(y_M-F_z^{-1}(\tilde{u}_{M-1})\right)\end{array}\right\}\leq 0\right\}.$$

Since

$$\max_{m=1,\ldots,M}y_m-F_z^{-1}(\tilde{u}_m)\leq F_{z+}^{-1}(\tilde{u}_{M+1})\leq y_M-F_z^{-1}(\tilde{u}_{M-1})$$
$$\Longrightarrow\max_{m=1,\ldots,M}y_m-F_z^{-1}(\tilde{u}_m)\leq y_M-F_z^{-1}(\tilde{u}_{M-1}),$$

it follows that

$$\mathcal{E}(\mathcal{Y})\subseteq\overline{\mathcal{E}}(\mathcal{Y}),$$

where

$$\overline{\mathcal{E}}(\mathcal{Y})\equiv\left\{\tilde{u}\in\mathcal{R}_{\tilde{U}}:\min_{y\in\mathcal{Y}}\max_{m=1,\ldots,M}y_m-y_M+F_z^{-1}(\tilde{u}_{M-1})-F_z^{-1}(\tilde{u}_m)\leq 0\right\}.$$

Furthermore, when $\tilde{u}\in\overline{\mathcal{E}}(\mathcal{Y})$ occurs, then there necessarily exists some v^+ such that

$$\max_{m=1,\ldots,M}y_m-F_z^{-1}(\tilde{u}_m)\leq v^+\leq y_M-F_z^{-1}(\tilde{u}_{M-1}),$$

and it likewise follows that there exists some strictly increasing CDF $F_{z+}(\cdot)$ such that for $\tilde{u}_{M+1}=F_{z+}(v^+)$

$$\max_{m=1,\ldots,M}y_m-F_z^{-1}(\tilde{u}_m)\leq F_{z+}^{-1}(\tilde{u}_{M+1})\leq y_M-F_z^{-1}(\tilde{u}_{M-1}).$$

The M inequalities appearing in $\overline{\mathcal{E}}(\mathcal{Y})$ can alternatively be obtained by differencing the inequalities delivered by the HT restrictions 1-2 appearing in (4.1). By definition $\tilde{U}$ is an element of the set defined in (4.1) if and only if

$$\forall m, \quad Y_m \leq F_z^{-1}\left(\tilde{U}_m\right) + F_{z+}^{-1}\left(\tilde{U}_{M+1}\right) \quad \wedge \quad Y_M \geq F_z^{-1}\left(\tilde{U}_{M-1}\right) + F_{z+}^{-1}\left(\tilde{U}_{M+1}\right),$$

since the combination of any such pair of inequalities for a given m implies

$$Y_m - Y_M \leq F_z^{-1}\left(\tilde{U}_m\right) - F_z^{-1}\left(\tilde{U}_{M-1}\right). \tag{4.3}$$

Looked at in this way the inequality appearing in (4.3) is obtained in similar manner to the derivation of observable implications in which fixed effects do not appear in panel data models. Here the auction-specific unobservable is akin to a fixed effect that appears in each of the inequalities delivered by restrictions 1-2. Combining these inequalities appropriately produces further observable implications from which the common unobservable term $v^+ = F_{z+}^{-1}(\tilde{u}_{M+1})$ is absent. The development here produces collections of such inequalities.

Defining

$$D \equiv (Y_M - Y_1, ..., Y_M - Y_{M-2}), \tag{4.4}$$

the inequalities (4.3) taken over all m can be written

$$\forall m = 1, ..., M-2: \quad D_m \geq F_z^{-1}\left(\tilde{U}_{M-1}\right) - F_z^{-1}\left(\tilde{U}_m\right).$$

Note that D only has $M-2$ components because (4.3) holds trivially for $m \in \{M-1, M\}$.

Let $\mathcal{D}$ denote a set of vectors on the support of random vector D. It follows that for any set $\mathcal{D} \subseteq \mathcal{R}_D$,

$$D \in \mathcal{D} \Rightarrow \left\{\exists d \in \mathcal{D} : \max_{m=1,\ldots,M-2} \left\{F_z^{-1}\left(\tilde{U}_{M-1}\right) - F_z^{-1}\left(\tilde{U}_m\right) - d_m\right\} \leq 0\right\}$$

and consequently

$$\mathbb{P}\left[D \in \mathcal{D} | z\right] \leq G_z^-\left(\widetilde{\mathcal{U}}\left(\mathcal{D}; F_z\right)\right), \tag{4.5}$$

where

$$\widetilde{\mathcal{U}}\left(\mathcal{D}; F_z\right) \equiv \left\{\tilde{u} \in \mathcal{R}_{\tilde{U}} : \min_{d \in \mathcal{D}} \max_{m=1,\ldots,M-2} F_z^{-1}\left(\tilde{u}_{M-1}\right) - F_z^{-1}\left(\tilde{u}_m\right) - d_m \leq 0\right\}.$$

The development using CR from which (4.5) was obtained allows us to establish that without additional restrictions placed on F_{z+}, these inequalities characterize sharp bounds on $\left(F_z(\cdot), G_z^-(\cdot)\right)$. The following theorem collects the formal results.

Theorem 3 *Let Restrictions 1-2, EX-V, EX-B, and UAH hold and let* $\mathcal{F} \in \mathsf{F}$, $\mathcal{F}_+ \in \mathsf{F}_+$, *and* $\mathcal{G} \in \mathsf{G}$. *Then* *(i) The identified set for* $\{F_z(\cdot), F_{z+}(\cdot), G_z(\cdot) : z \in \mathcal{R}_z\}$ *are those admitted by* $\mathsf{F}, \mathsf{F}_+, \mathsf{G}$ *that satisfy, for almost every* $z \in \mathcal{R}_z$*:*

$$\forall \mathcal{Y} \subseteq \mathcal{R}_Y, \quad \mathbb{P}[Y \in \mathcal{Y}|z] \leq G_z(\mathcal{E}(\mathcal{Y})).$$

(ii) With no restrictions placed on F_+*, the identified set for* $\{F_z(\cdot), G_z^-(\cdot) : z \in \mathcal{R}_z\}$ *are those admitted by* F, G *that satisfy, for almost every* $z \in \mathcal{R}_z$*:*

$$\forall \mathcal{D} \subseteq \mathcal{R}_D, \quad \mathbb{P}[D \in \mathcal{D}|z] \leq G_z^-\left(\widetilde{\mathcal{U}}(\mathcal{D}; F_z)\right). \tag{4.6}$$

(iii) With no restrictions placed on F_+*, the identified set for* $\{F_z(\cdot) : z \in R_z\}$ *are those admitted by* F *such that for some* $G_z^-(\cdot)$ *admitted by* G*, (4.6) holds. If, in addition, Restriction IPV holds, then* $\left(\tilde{U}_1, ..., \tilde{U}_M\right)$ *is distributed independently of* Z *such that* $G_z^-(\cdot)$ *corresponds to the joint distribution of the order statistics of* $M-1$ *independent uniform*$(0,1)$ *variables, and for any* $F_z(\cdot)$ *the probability on the right of (4.6) is known.*

In addition to characterizing identified sets under the stated restrictions, Theorem 3 provides a starting point for examining further simplifications of these characterizations that may be attainable under particular restrictions on F, F_+, and G. For example, the theorem allows for, but requires neither independence of observed characteristics Z and latent auction heterogeneity V^+, nor independence between V^+ and V^I. Such restrictions may allow further simplification of these characterizations.

5 Observable Auction Specific Heterogeneity

In this Section we show how information on the effect of observable auction characteristics on valuations can be extracted from English auction data. There is good reason to be interested in the impact of observable auction characteristics on bidders' valuations. The survivor function of the distribution of valuations is effectively the demand function faced by the seller and the coefficients β inform us about the sensitivity of demand to variations in the characteristics of the item for sale.

We now introduce an index restriction on the effect of *observable* auction heterogeneity on individual valuations. The analysis of this Section delivers inequalities that define an outer region for the index coefficients which apply under general forms of departure from IPV.

Restriction IR (Index Restriction). The M bidder valuations are given by $V = (V_1, ..., V_M)$ where for each $j = 1, ..., M$,

$$V_j = X\beta + V_j^I + V^+.$$

Conditional on the number of bidders $M = m$, $(V_1^I, ..., V_M^I, V^+)$ are independent of X with $V_j^I \sim F_m(\cdot)$, each $j = 1, ..., m$, and $V^+ \sim F_{m+}(\cdot)$, where $F_m(\cdot)$ and $F_{m+}(\cdot)$ are both strictly increasing on their supports.

This restriction refines Restriction UAH by specifying how observable auction characteristics X affect the distribution of bidder valuations, requiring that they do so only through the linear index $X\beta$, conditional on the number of bidders. The joint distribution of the components of bidder valuations may nonetheless be jointly dependent, and may vary with the number of bidders M.

Furthermore, the index restriction on bidder valuations in Restriction IR does not imply an index restriction on the dependence on X of the joint distribution of *bids*. This is because bidding strategies may depend on X in arbitrary ways – not just through the linear index $X\beta$ – even when valuations satisfy the single index Restriction IR. So the bounds below apply when bidding strategies vary with auction characteristics, and in arbitrary ways subject to Restrictions 1-2 and IR holding. Since X can affect the distribution of bids not only through the index $X\beta$ conventional single index estimation methods applied to final bid data would not be justified.

Let

$$\overline{V}^{+} \equiv \left(V_1^I + V^+, ..., V_M^I + V^+\right),$$

and let $\tilde{V}^+$ denote the order statistics of $\overline{V}^+$, so that

$$\tilde{V}^{+} \equiv \left(V_{1:M}^I + V^+, ..., V_{M:M}^I + V^+\right).$$

The HT restrictions on bidder behavior remain that for all $j = 1, ..., M$, $V_j \geq B_j$ and $V_{M-1:M} \leq B_{M:M}$. The (unordered) set of $\bar{V}^+$ permissible given observed bids B and $Z = (X, M)$ is

$$\mathcal{V}_\beta\left(B, Z\right) = \left\{\overline{v}^+ : \forall j,\, \overline{v}_j^+ \geq B_j - X\beta \wedge \overline{v}_{M-1:M}^+ \leq B_{M:M} - X\beta\right\}. \tag{5.1}$$

The (ordered) set of $\tilde{V}^+$ permissible given observed ordered bids $Y = (B_{1:M}, ..., B_{M:M})$ and Z is

$$\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) = \left\{\tilde{v}^+ : \forall j,\, \tilde{v}_j^+ \geq Y_j - X\beta \wedge \tilde{v}_{M-1}^+ \leq Y_M - X\beta\right\}, \tag{5.2}$$

it being understood that $\tilde{v}_1^+ \leq \cdots \leq \tilde{v}_M^+$. From here on we work with $\widetilde{\mathcal{V}}_\beta\left(Y, Z\right)$ rather than $\mathcal{V}_\beta\left(B, Z\right)$.

For each $m \in \mathcal{M} \equiv \left\{2, ..., \overline{M}\right\}$, let $\tilde{G}_V\left(\mathcal{S}|m\right)$ denote the conditional probability that $\tilde{V}^+$ is in $\mathcal{S}$ given $M = m$. The identified set for $\left(\beta, \tilde{\mathcal{G}}_V\right)$ can be represented as those that satisfy

$$\forall m \in \mathcal{M},\, \forall \mathcal{S} \in \mathsf{Q}_m, \quad \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}|x, m\right] \leq \tilde{G}_V\left(\mathcal{S}|m\right) \text{ a.e. } x \in \mathcal{R}_{X|m}, \tag{5.3}$$

equivalently

$$\forall m \in \mathcal{M},\, \forall \mathcal{S} \in \mathsf{Q}_m, \quad \sup_{x \in \mathcal{R}_{X|m}} \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}|x, m\right] \leq \tilde{G}_V\left(\mathcal{S}|m\right),$$

where Q_m denotes unions of sets of the form given in (5.2). A class of sets which is guaranteed to contain all such sets, and which depends neither on observed variables nor on β is given by the collection S_m defined as

$$\mathsf{S}_m \equiv \left\{ \mathcal{S}\left(\mathcal{A}_m\right) : \mathcal{A}_m \subseteq \mathcal{R}_{\tilde{V}^+|m} \right\},$$

where $\mathcal{R}_{\tilde{V}^+|m}$ denotes the conditional support of $\tilde{V}^+$ given $M = m$ and

$$\mathcal{S}\left(\mathcal{A}_m\right) \equiv \bigcup_{a \in \mathcal{A}_m} \left\{ \tilde{v}^+ \in \mathcal{R}_{\tilde{V}^+|m} : \forall k = 1, ..., m,\ \tilde{v}_k^+ \geq a_k \wedge \tilde{v}_{m-1}^+ \leq a_m \right\}.$$

Thus we can write the identified set for $\left(\beta, \tilde{\mathcal{G}}_V\right)$ as those that satisfy

$$\forall m \in \mathcal{M},\ \forall \mathcal{S} \in \mathsf{S}_m, \quad \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}|x, m\right] \leq \tilde{G}_V\left(\mathcal{S}|m\right) \text{ a.e. } x \in \mathcal{R}_{x|m}. \tag{5.4}$$

Note that we could remove from the collection of test sets S_m those which can be written as unions of disjoint sets. Two sets $\mathcal{S}\left(\{a\}\right)$ and $\mathcal{S}\left(\{a'\}\right)$ are disjoint if either $a'_{m-1} \geq a_m$ or $a_{m-1} \geq a'_m$.

Because the dimension of $\tilde{V}^+$ depends on M, the mapping $\tilde{G}_V\left(\cdot|m\right)$ depends on m. This is so even if valuations are additionally assumed independent of M. Nonetheless, for each fixed m Corollary 3 of CR can be applied to produce bounds on the parameter β, since the distribution of $\tilde{V}^+$ conditional on $M = m$ and $X = x$ does not vary with x. The resulting bounds are given by the set of values of β that satisfy

$$\forall m \in \mathcal{M},\ \forall \mathcal{S} \in \mathsf{S}_m, \quad \sup_{x \in \mathcal{R}_{X|m}} \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}|x, m\right] \leq \inf_{x \in \mathcal{R}_{X|m}} \left(1 - \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}^c|x, m\right]\right). \tag{5.5}$$

These bounds can be used to place limits on the relative impact of the components of observable auction characteristics X on bidder valuations as we demonstrate in Section 6.

5.1 Containments and Capacities

In consideration of inequalities of the form (5.5), consider a test set $\mathcal{S} \equiv \mathcal{S}\left(\mathcal{A}_m\right)$ for a given m. The *containment* functional is

$$\begin{aligned} \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta\left(Y, Z\right) \subseteq \mathcal{S}\left(\mathcal{A}_m\right)|z\right] &= \mathbb{P}\left[\exists a \in \mathcal{A}_m\colon\ \forall k \leq m,\ Y_k - x\beta \geq a_k \wedge Y_{m-1} - x\beta \leq a_m|z\right] \\ &= \mathbb{P}\left[\min_{a \in \mathcal{A}_m} \max\left\{Y_{m-1} - x\beta - a_m, \max_{k \leq m} a_k - Y_k + x\beta\right\} \leq 0|z\right]. \end{aligned}$$

The *capacity* functional for a test set $\mathcal{S} = \mathcal{S}(\mathcal{A}_m)$ is

$$\begin{aligned} 1 - \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta(Y,Z) \subseteq \mathcal{S}(\mathcal{A}_m)^c \,|z\right] &= \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta(Y,Z) \cap \mathcal{S}(\mathcal{A}_m) \neq \emptyset |z\right] \\ &= \mathbb{P}\left[\min_{a\in\mathcal{A}_m} \{\max\{a_{m-1}, Y_{m-1} - x\beta\} - \min\{a_m, Y_m - x\beta\}\} \leq 0|z\right] \end{aligned}$$

When bids are continuously distributed sets $\mathcal{S}(\mathcal{A}_m)$ will be required to comprise what we term contiguous unions of level-sets $\widetilde{\mathcal{V}}_\beta(Y,Z)$ – or unions of such contiguous unions – in order for the containment probability to be greater than zero. A contiguous union is characterized by a vector $a = (a_1, ..., a_m, a'_m)$ with $a_1 \leq \cdots \leq a_m \leq a'_m$, with the contiguous union given by the union of sets $\widetilde{\mathcal{V}}_\beta(y,z)$ for all m-vectors y such that $y_j = a_j$ for all $j < m$ and y_m takes all values in the interval $[a_m, a'_m]$. In other words, this is the union of level sets $\widetilde{\mathcal{V}}_\beta(y,z)$ taken as the first $m-1$ components of y are held fixed and the highest is moved over the range $[a_m, a'_m]$.

Such a contiguous union may be represented as

$$\mathcal{S}(a) \equiv \left\{\tilde{v}^+ : \forall j \quad \tilde{v}_j^+ \geq a_j \wedge \tilde{v}_{m-1}^+ \leq a'_m\right\},$$

for any vector $a \in \mathbb{R}^{m+1}$ whose components are ordered from smallest to largest. Likewise, it is to be understood that the components of $\tilde{v}^+ \in \mathbb{R}^m$ are ordered such that $\tilde{v}_1^+ \leq \cdots \leq \tilde{v}_m^+$.

The conditional containment functional for such a contiguous union is

$$\mathbb{P}\left[\widetilde{\mathcal{V}}_\beta(Y,Z) \subseteq \mathcal{S}(a)\,|x,m\right] = \mathbb{P}\left[\wedge_{j=1}^m \{Y_j - X\beta \geq a_j\} \wedge \{Y_m - X\beta \leq a'_m\}\,|x,m\right].$$

The corresponding conditional capacity functional is

$$1 - \mathbb{P}\left[\widetilde{\mathcal{V}}_\beta(Y,Z) \subseteq \mathcal{S}(a)^c\,|z\right] = \mathbb{P}\left[\{Y_{m-1} - X\beta \leq a'_m\} \wedge \{Y_m - X\beta \geq a_{m-1}\}\,|x,m\right].$$

And so (5.5) applied to such contiguous unions produce the inequalities

$$\begin{aligned} &\sup_{x\in\mathcal{R}_{x|m}} \mathbb{P}\left[\wedge_{j=1}^m \{Y_j - X\beta \geq a_j\} \wedge \{Y_m - X\beta \leq a'_m\}\,|x,m\right] \\ &\qquad \leq \inf_{x\in\mathcal{R}_{x|m}} \mathbb{P}\left[\{Y_{m-1} - X\beta \leq a'_m\} \wedge \{Y_m - X\beta \geq a_{m-1}\}\,|x,m\right] \quad (5.6) \end{aligned}$$

which must hold for possible numbers of bidders $m \in \mathcal{M}$ and for all $a_1 \leq \cdots \leq a_m \leq a'_m$.

6 Illustrative calculations

In this section we specify a particular class of structures for generating final bids made by $M = 3$ bidders which embodies auction-specific observed heterogeneity X which enters via a linear index as set out in Restriction IR. Idiosyncratic elements of valuations are allowed to be correlated and bidding strategies can be X-dependent.

We calculate outer regions for index coefficients using various selections of the inequalities given in (5.6). The probabilities in those inequalities are calculated as relative frequencies of the occurrence of the required events at each value of X observed in $10,000$ simulated auctions. We take this approach because calculation by other means of probabilities in the complex auction mechanism considered is infeasible. We find that the outer regions are quite informative and respond as expected to changes in the choice of inequalities employed.

The specification of bidders' valuations is set out in Section 6.1 after which the auction mechanism is described in Section 6.2. The calculation of outer regions using grid search is described in Section 6.3.

6.1 Valuations

The log valuation of bidder j amongst M bidders is denoted V_j and determined as

$$V_j = X\beta + V_j^I + V^+,$$

where $V^I \equiv (V_1^I, \ldots, V_M^I)$, V^+, and $X \equiv (X_1, X_2)$ are mutually independently distributed. The elements of V^I are distributed $N(0, \Sigma)$ where

$$\Sigma = \begin{bmatrix} 1 & 0.2 & 0.5 \\ 0.2 & 1 & 0.4 \\ 0.5 & 0.4 & 1 \end{bmatrix}$$

and scalar V^+ is distributed $N(0,1)$. Consequently $\left(V_1^I + V^+, ..., V_M^I + V^+\right)$ is independent of X, as required by Restriction IR, and is distributed $N(0, \Omega)$ where

$$\Omega = \begin{bmatrix} 2 & 1.2 & 1.5 \\ 1.2 & 2 & 1.4 \\ 1.5 & 1.4 & 2 \end{bmatrix}.$$

Each element of X has support on $\{-1, 0, 1\}$, each value $(x_1, x_2) \in \{-1, 0, 1\} \times \{-1, 0, 1\}$ occurs with probability $1/9$ and $\beta = (-1, 1)$.

The distribution of $X\beta$ is symmetric around zero with support $\{-2, -1, 0, 1, 2\}$ and with probability $2/9$ on $|X\beta| = 2$ and probability $4/9$ on $|X\beta| = 1$. The variance of $X\beta$ is $1\frac{1}{3}$. The variance of the unobserved elements of each log valuation is 2.

Table 1: Quantiles of random bidding fractions at values of X

x_1	x_2	10th percentile	median	90th percentile
-1	-1	.20	.50	.80
-1	0	.14	.39	.68
-1	1	.11	.31	.58
0	-1	.32	.61	.86
0	0	.25	.50	.75
0	1	.20	.42	.67
1	-1	.42	.69	.89
1	0	.33	.58	.80
1	1	.28	.50	.72

6.2 Auction mechanism

Each bidder j of the M bidders is assigned a valuation, V_j, as above and also the value λ_j of a random bidding fraction, Λ_j. Bidding fractions vary across auctions in a manner determined by the value of the auction specific characteristics X.

Bidding fractions are independent across bidders and independent of the components of valuations and when $X = x$ they are realizations of $Beta(\gamma_1(x), \gamma_2(x))$ random variables where

$$\begin{aligned} \gamma_1(x) &= 3 + x_1 \\ \gamma_2(x) &= 3 + x_2. \end{aligned}$$

These Beta distributed random variables have support on $[0, 1]$, with median and 10^{th} and 90^{th} percentiles given $X = x$ as shown in Table 1.

At each round i, bidder j is chosen from the remaining eligible bidders. The eligible bidders are those with valuations exceeding the bid on the table, b, zero in the first round, excluding the bidder who placed the bid on the table. At round i the chosen bidder, j, bids the amount $\Lambda_j V_j + (1 - \Lambda_j)b$. The auction proceeds until there remains one bidder. The mechanism results in final bids that satisfy the HT conditions.

6.3 Calculations using grid search

We simulate $10,000$ auctions and calculate the conditional on $X = x$ probabilities in (5.6) as relative frequencies of occurrence of the events indicated amongst the simulated auctions for each value x under consideration.

Inequalities as given in (5.6) are determined by $M + 1$ ordered values on the extended real line, $(a_1, \ldots, a_M, a'_M)$, which defines a contiguous union of U-level sets. In the calculation of a particular outer region reported here the inequalities employed are obtained using the contiguous

Table 2: Rounded values of log valuations in lists used to define contiguous unions

$L(6)$	$L(11)$
$-\infty$	$-\infty$
-1.89	-3.20
-0.59	-1.89
$+0.40$	-1.16
$+1.50$	-0.59
$+\infty$	-0.10
	$+0.40$
	$+0.90$
	$+1.50$
	$+2.32$
	$+\infty$

Table 3: Projections of identified sets obtained from grid search on a 50*50 grid

$L(x)$	# inequalities	β_1^L	β_1^U	β_2^L	β_2^U
$L(6)$	53	-1.263	-0.777	0.801	1.268
$L(11)$	253	-1.251	-0.812	0.833	1.226

unions delivered by all admissible selections from a list of x values $L(x)$ containing at most d distinct finite values with $1 \leq d \leq M + 1$. In all the calculations reported here $d = 2$.

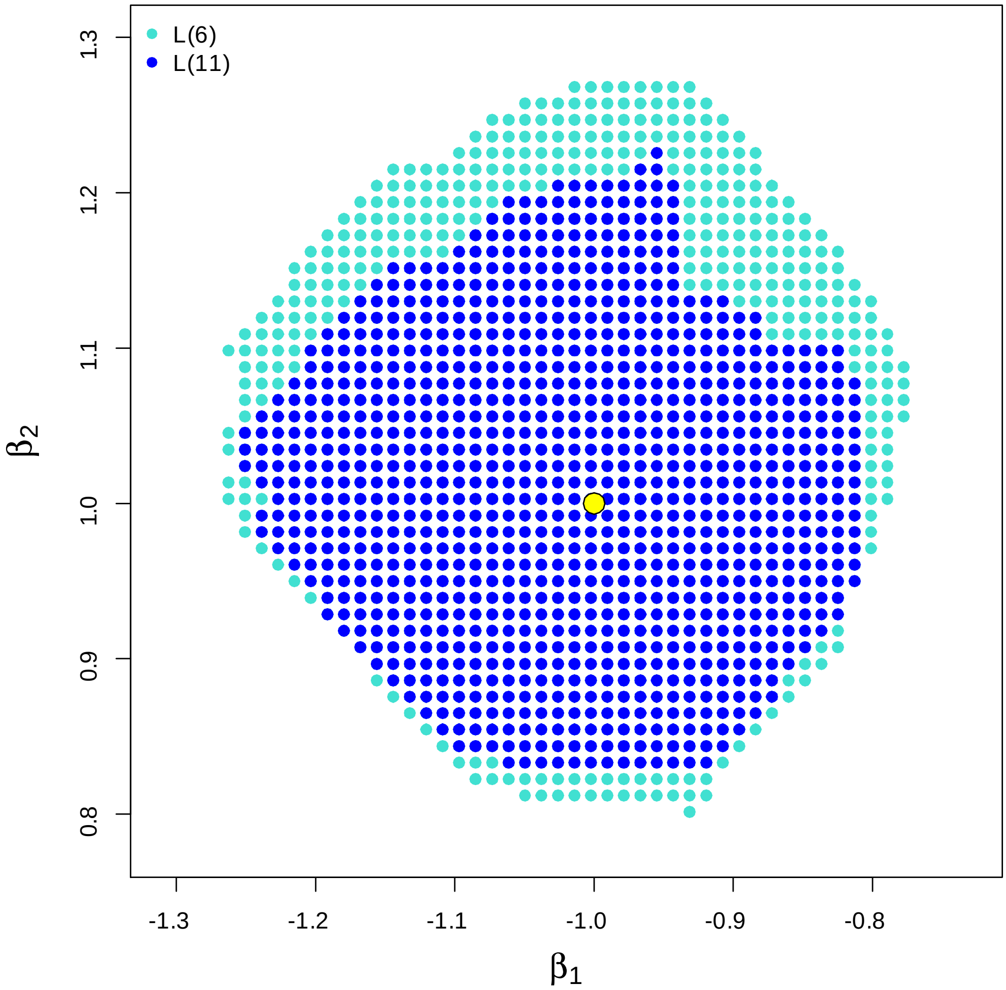
Two lists are considered. One with 6 elements, $L(6)$, comprises $-\infty$ and $+\infty$ and the $(0.2, 0.4, 0.6, 0.8)$ quantiles of the log of highest and second highest simulated final bids. The other, with 11 elements, $L(11)$ comprises $-\infty$ and $+\infty$ and the $(0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$ quantiles of the log of highest and second highest simulated final bids. The lists are shown in Table 2. The values in list $L(6)$ all appear in the list $L(11)$ so the identified sets obtained using $L(11)$ are subsets of the identified sets obtained using $L(6)$.

Grid search was conducted with each list of x values, $L(6)$ and $L(11)$. The resulting outer regions are shown in Figure 1. Table 3 shows projections of the identified sets onto the β_1 and β_2 axes. Moving from 53 to 253 inequalities results in a significant reduction in the identified set.

7 Concluding remarks

The characterizations using the CR Generalized Instrumental Variable model development open the door to the use of Restrictions 1-2 in auction models that do not require independent private values. These restrictions on bidder behavior, introduced by HT, are intuitively appealing in open outcry auctions where the usual button auction equilibrium may not be an appropriate model of bidder behavior. HT showed that even though these restrictions may seem a strong relaxation of

Figure 1: Identified sets obtained by search on a 50×50 grid. Points in the identified set using $L(11)$ shown in blue. Points in the identified set using $L(6)$ shown in turquoise and blue. The value of (β_1, β_2) in the structure generating final bids is $(-1, 1)$, marked in yellow.



the restriction that bidders play equilibrium strategies, and they render the model incomplete, they can still be used to learn useful information about valuation distributions in a model with IPV. In many auctions studied in empirical research, the IPV paradigm may be questionable, and this has sometimes motivated the use of auction models that allow for private values. In Theorems 1, 2, and 3 characterizations of identified sets for model primitives were developed in model that do not require IPV. Theorem 3 in particular applies to models that allow for affiliated private values through auction-specific unobservable heterogeneity, which has been a focus of some recent papers in the literature, see for example Krasnokutskaya (2011), Armstrong (2013), Roberts (2013), and Quint (2015).

Bounds were additionally developed on the effect of observable auction characteristics on bidder valuations in a model incorporating a familiar index restriction. Numerical illustrations of these bounds based on simulated auction models were presented, illustrating the potential for the bounds to be informative. In the case considered in which covariate coefficients were 1 and -1 the identified set pinned down the values of these coefficients to within plus or minus 20%.

The bound characterizations and identified sets for these English auction models involve a dense system of inequalities. In CR it was demonstrated in the IPV setting that these inequalities restrict not only the level of the bidder valuation distribution function at each point on its support but also the shape of the function as it passes between the pointwise bounds. The richness of the collection of inequalities characterizing identified sets in non IPV setting is also potentially informative about model primitives and functionals of these in models in which IPV is not assumed. The application of these inequalities to perform inference using real world auction data is a direction in which we are continuing to work.

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A Proofs of results stated in the main text

Proof of Lemma 1. Consider a realization (b,v) of (B,V). Under Restriction 1 the number of elements of b with values greater than $v_{m:M}$ is at most $M-m$. Therefore in all realizations of (B,V), $b_{m:M} \leq v_{m:M}$ for all m and M, from which (3.6) follows immediately. The second result, (3.7), follows directly from Restriction 2. □

Proof of Theorem 1. The result follows from application of Corollary 1 of Theorem 2 and Lemma 1 in CR with U-sets $\mathcal{U}(B,F_z)$ as defined in (3.2). □

Proof of Theorem 2. From Theorem 1 we have that the identified set for $\{F_z(\cdot), G_z(\cdot) : z \in \mathcal{R}_Z\}$ are those such that

$$\mathbb{P}\left[\mathcal{U}(B,F_z) \subseteq \mathcal{S} | Z = z\right] \leq G_z(\mathcal{S}) \tag{A.1}$$

for all $\mathcal{S}$ comprising unions of sets on the support of $\mathcal{U}(B,F_z)$, a.e. $z \in \mathcal{R}_Z$. From CR Lemma 1 this is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_W), \quad \mathbb{P}\left[\mathcal{U}(B,F_z) \subseteq \mathcal{S} | Z\right] \leq G_z(\mathcal{S}) \tag{A.2}$$

where $\mathsf{C}(\mathcal{R}_W)$ denotes the collection of closed sets $\mathcal{S}$ on $[0,1]^M$, the support of U.

For the purpose of the proof, fix $z \in \mathcal{R}_Z$ at an arbitrary value. All probability statements below are to be understood to be conditional on $Z = z$.

Let W be a random M vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$ distributed $G_z(\cdot)$ and independent of B. Then (A.2) is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_W), \quad \mathbb{P}\left[\mathcal{U}(B,F_z) \subseteq \mathcal{S}\right] \leq \mathbb{P}\left[W \in \mathcal{S}\right]. \tag{A.3}$$

Let B^0 denote that random M-vector whose components are precisely those of B, reordered in such a way that for each $m = 1, ..., M$, the m^{th} lowest component has the same index as that of the m^{th} lowest component of W. B^0 is therefore a random permutation of B, and since W and B are independent, exchangeability of B implies that B^0 has the same distribution as B.

We consequently have that (A.3) holds if and only if

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_U), \quad \mathbb{P}\left[\mathcal{U}\left(B^0,F_z\right) \subseteq \mathcal{S}\right] \leq \mathbb{P}\left[W \in \mathcal{S}\right]. \tag{A.4}$$

Equivalently, by Artstein's inequality we have that the distribution of W is selectionable with respect to the distribution of $\mathcal{U}\left(B^0,F_z\right)$, such that there exist random vectors $W^* \overset{d}{=} W$ ($\overset{d}{=} \tilde{U}$) and $B^* \overset{d}{=} B^0$ ($\overset{d}{=} B$) such that $\mathbb{P}\left[W^* \in \mathcal{U}(B^*,F_z)\right] = 1$.[12] By definition of the set-valued mapping

[12] Artstein's inequality is from Artstein (1983), see also Norberg (1992) and Molchanov (2005, Section 1.4.8).

$\mathcal{U}(\cdot, F_z)$, we therefore have that (A.4) is equivalent to the statement that with probability one

$$\forall m,\ F_z(B_m^*) \leq W_m^* \wedge F_z\left(B_{m(B^*)}^*\right) \geq \max_{m \neq m(B^*)} W_m^*.$$

Because the elements of B^* and W^* have the same ordering, this holds if and only if with probability one

$$\forall m,\ F_z(B_{m:M}^*) \leq W_{m:M}^* \wedge F_z(B_{M:M}^*) \geq W_{M-1:M}^*,$$

or equivalently, since $B^* \stackrel{d}{=} B$, $Y = (B_{1:M}, ..., B_{M:M})$, and $W^* \sim G_z$, if and only if the distribution of $\tilde{U} = (U_{1:M}, ..., U_{M:M})$ is selectionable with respect to the distribution of

$$\widetilde{\mathcal{U}}(Y, F_z) \equiv \left\{u \in [0,1]^M : u_1 \leq \cdots \leq u_M \wedge \forall m,\ u_m \geq F_z(Y_m) \wedge u_{M-1} \leq F_z(Y_M)\right\}.$$

The distribution of $\tilde{U}$ is that of the order statistics of U, such that for any set $\mathcal{S}$, $\mathbb{P}\left[\tilde{U} \in \mathcal{S}\right] = \widetilde{G}_z(\mathcal{S})$, so that by application of Artstein's (1983) inequality, the above selectionability condition is equivalent to

$$\forall \mathcal{S} \in \mathsf{C}(\mathcal{R}_U), \quad \mathbb{P}\left[\widetilde{\mathcal{U}}(Y, F_z) \subseteq \mathcal{S} | z\right] \leq \widetilde{G}_z(\mathcal{S}).$$

Since the choice of z was arbitrary, this concludes the proof. □

Proof of Theorem 3. Part (i) follows from application of Theorem 2 as described in the main text. Part (ii) follows from first noticing that CR Corollary 1 and Lemma 1 in conjunction with the definition of random vector D imply that the set of $(F_z(\cdot), G_z^-(\cdot))$ satisfying (4.6) are precisely those such that conditional on $Z = z$ there exist random vectors $\tilde{Y} \stackrel{d}{=} Y | Z = z$ and $\bar{U} \sim G_z^-(\cdot)$ satisfying (4.3) with each Y_m replaced by $\tilde{Y}_m$ and $\tilde{U}$ replaced by $\bar{U}$. This guarantees that for all m,

$$\tilde{Y}_m - F_z^{-1}\left(\bar{U}_m\right) \leq \tilde{Y}_M - F_z^{-1}\left(\bar{U}_{M-1}\right),$$

and so there exists a random variable $\tilde{V}^+$ such that with probability one

$$\tilde{Y}_m - F_z^{-1}\left(\bar{U}_m\right) \leq \tilde{V}^+ \leq \tilde{Y}_M - F_z^{-1}\left(\bar{U}_{M-1}\right).$$

Thus we have established the existence of random vectors $\tilde{Y} \stackrel{d}{=} Y | Z = z$ and $\bar{U} \sim G_z^-(\cdot)$ such that Restrictions 1-2 and UAH hold. Part(iii) follows from taking the implied set of feasible F_z from part (ii) and that with Restriction IPV, $G_z^-(\cdot)$ is known. □

Figure 2: $\tilde{U}$ level sets in a 2 bidder case. The level set $\widetilde{\mathcal{U}}((y_1, y_2), F_z)$, shaded in the Figure, contains all values of uniform order statistics, $\tilde{u}_2 \geq \tilde{u}_1$, that can give rise to order statistics of final bids, $y_2 \geq y_1$ such that, $F_z(y_1) = 0.2$ and $F_z(y_2) = 0.6$. F_z is a potential distribution function of valuations. For fixed values y_1 and y_2 changing F_z would change the numerical values 0.2 and 0.6.