Instrument-Free Demand Estimation: Methodology and an Application to Consumer Products*

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Abstract

We consider identification and estimation of demand systems in models of imperfect competition. When the standard assumption of profit maximization is imposed, the resulting supply-side restrictions allow us to correct for price endogeneity bias without the use of instruments. We show that the biased coefficient from an ordinary least squares regression of (transformed) quantity on price can be expressed as function of demand parameters. When combined with minimal assumptions about the correlations among unobservable shocks, the profit-maximization conditions are sufficient to identify the key parameters of several models commonly used in demand estimation. We illustrate the methodology with applications to the cement industry and the airline industry.

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1 Introduction

A central challenge of demand estimation is price endogeneity. If prices reflect demand shocks that are not observed by the econometrician, then an ordinary least squares regression (OLS) will not recover the casual demand curve (Working (1927)). In this paper, we reconsider whether exogenous variation in prices is necessary to recover causal demand parameters. Starting from the standard premise that firms set prices to maximize profits, we show that the supply-side assumptions already maintained in many structural models dictate how prices respond to demand shocks. It is possible to correct for endogeneity bias without relying on instruments by leveraging these assumptions in estimation. Consistent estimation of empirical models has been a major focus of modern empirical and econometric research in industrial organization (e.g., Berry et al. (1995); Berry and Haile (2014)), which has, thus far, relied heavily on the search for instruments.

The methodology we introduce begins with an analysis of observed equilibrium variation in prices and (possibly transformed) quantities, as summarized by OLS regression. We show that, for many standard empirical models of imperfect competition, the bias that arises with OLS can be quantified as a function of the data. Thus, OLS estimates are informative, as they capture a blend of the demand curve and the endogenous response by firms. Supply-side assumptions may be used to construct bounds on the structural parameters and, with the addition of a surprisingly weak assumption, achieve point identification. Interpreted through the lens of a sufficiently tight model, the causal relationships captured by OLS can be isolated and recovered. The methodology essentially uses economic theory as a substitute for exogenous variation in prices, allowing for consistent estimates of structural parameters without the use of instruments.

The surprisingly weak assumption needed for point identification relates to the covariance between the unobserved demand and marginal cost shocks in the model. If the econometrician has prior knowledge of this covariance, then typically the price parameter is point identified. A reasonable assumption in some settings is *uncorrelatedness*, i.e., that there is no covariance between unobserved shocks. The identifying power of covariance restrictions in the context of linear systems was pursued in early Cowles Foundation research (Koopmans et al., 1950). Despite considerable methodological advances, including to semi-parametric and non-parametric systems (Hausman and Taylor (1983); Matzkin (2016); Chiappori et al. (2017)), we are not aware of any previous papers that examine the identifying power of these restrictions in models of imperfect competition.

We develop intuition using a model of a monopolist with constant marginal costs and linear demand (Section 2). Equilibrium variation in prices and quantities (p and q) is generated by uncorrelated demand and cost shocks (ξ and η) that are unobservable to the econometrician. We prove that consistent estimate of the price parameter, β, is given by

$$\hat{\beta} = -\sqrt{\left(\hat{\beta}^{OLS}\right)^2 + \frac{Cov(\hat{\xi}^{OLS}, q)}{Var(p)}}$$

where $\hat{\beta}^{OLS}$ is the price coefficient from an OLS regression of quantities on prices, and $\hat{\xi}^{OLS}$ is a vector of the OLS residuals. The information provided by OLS regression is sufficient for the consistent identification of the structural parameter. This holds whether variation arises predominately from demand shocks or from supply shocks—economic theory allows for the recovery of the causal parameter regardless of the shape of the well-known "cloud" of price-quantity pairs. Our results in this section map cleanly to the textbook treatment of the simultaneous equations bias that arises with supply and demand (e.g., Hayashi 2000, chapter 3).

In Section 3, we obtain our main result under two common assumptions about demand and supply. We assume that demand is semi-linear in prices after a known transformation. This assumption nests many differentiated-products demand systems, including the random coefficients logit (e.g., Berry et al. (1995)) and the market-specific demand curves that often are applied to homogeneous products. On the supply side, we start with the assumption that firms compete in prices and have constant marginal costs. We prove that the price parameter, β, solves a quadratic equation in which the coefficients are functions of the data and $Cov(\xi, \eta)$. We provide a verifiable sufficient condition under which β is the lower root of the quadratic; if the condition holds then knowledge of $Cov(\xi, \eta)$ point identifies β. We derive a consistent "three-step" estimator from the quadratic formula that takes the OLS coefficient and residuals as inputs. Alternatively, estimation can be conducted with the method of moments. Monte Carlo evidence confirms that these estimators perform well in small samples. Without exact knowledge of of $Cov(\xi, \eta)$, it possible to bound β using appropriate priors, such as $Cov(\xi, \eta) \geq 0$. Notably, the model structure may dictate that some values of $Cov(\xi, \eta)$ cannot reconcile the data with the model, therefore generating prior-free bounds on β. The model-implied bounds can be combined with a prior over $Cov(\xi, \eta)$ to narrow the identified set.

Though the above assumptions about demand and supply are not uncommon in applications, they are relatively strong restrictions that may be incompatible with features of the true data-generating process. In Section 4, we discuss three such assumptions in detail. First, if marginal costs are non-constant, then unobserved demand shocks affect prices both through markup adjustments and marginal costs. We prove that the identification of β is preserved if the non-constant portion of marginal costs can be brought into the model and estimated. Otherwise, it is more appropriate to proceed with a bounds approach under an assumption such as $Cov(\xi, \eta) \geq 0$. Second, we show that our insight and our approach are not dependent on the precise nature of the competitive game, but instead is driven by a general result that prices can be written as costs plus a markup term. Third, we examine the uncorrelatedness assumption itself. Of course, there are a variety of reasons that marginal costs may be correlated with demand unobservables. In our view, uncorrelatedness becomes relatively more palatable if these connections can be brought into the model explicitly. In applications with panel data, fixed effects may be used to absorb product- and time-specific characteristics, thus eliminating the key sources of correlation in unobservables in many settings.

To demonstrate how to apply the methodology, we conduct two empirical exercises in Section 5. The first exercise examines the cement industry using the model and data of Fowlie et al. (2016) ["FRR"]. We establish that identification extends to Cournot competition with homogeneous products, and we find that our three-step estimator produces a similar elasticity to that obtained by two-stage least squares (2SLS) estimation using the instruments of FRR. To implement our uncorrelatedness-based estimator, we take as given the shape of FRR's "hockey-stick" marginal cost function, which incorporates capacity constraints in production. Because (positive) demand shocks induce firms to produce at higher marginal cost, leaving the "hockey stick" cost component as unobserved would cause the uncorrelatedness assumption to fail. However, we show that the alternative assumption $Cov(\xi, \eta) \geq 0$ is sufficient to bound the price elasticity well away from the OLS estimate.

The second empirical exercise examines the airlines industry using the model and data of Aguirregabiria and Ho (2012) ["AH"]. Demand is specified as nested logit. We find that for relatively high values of the nesting parameter, σ, smaller (less negative) values of β cannot reconcile the data with the model, and this produces an exclusive bound on β that easily rejects the OLS estimate without requiring any stance on uncorrelatedness. We then jointly estimate σ and β assuming uncorrelatedness. This requires at least one supplemental moment. Natural candidates include across-nest correlation restrictions and higher-order restrictions, either of which allow for the joint identification of the (β, σ) parameters without instruments. We find that both 2SLS with the AH instruments and estimation using covariance restrictions move the parameters in the expected direction relative to OLS.

Our final contribution is an original application to retail scanner data. [**Coming soon.**]

Our research builds on a number of literatures in economics. As mentioned above, the identifying power of covariance restrictions in linear systems of equations was recognized in early work at the Cowles Foundation (Koopmans et al. (1950)). Subsequently, Hausman and Taylor (1983) proposed a two-stage approach for the estimation of supply and demand models of perfect competition: First, the supply equation is estimated with 2SLS using an instrument taken from the demand-side of the model. Second, the supply-side error term is recovered and, under the assumption of uncorrelatedness ($Cov(\xi, \eta) = 0$), it serves as a valid instrument for the estimation of demand. Matzkin (2016) extends this "unobserved instrument" approach to semi-parametric models; see also the contemporaneous research of Chiappori et al. (2017). Market power is not considered in these articles because it would imply that the supply-side equations depend on demand parameters through unobserved firm markups—precluding the recovery of unobserved costs via the first-stage regression. Our methodology deals with imperfect competition by estimating supply and demand *jointly* and allowing economic theory to determine the unobserved markups.

The identification challenge we address has been a major focus of modern empirical and econometric research in industrial organization. Typically the challenge is case as a problem of finding valid instruments. Many possibilities have been developed, including the attributes of competing products (Berry et al. (1995); Gandhi and Houde (2015)), the prices of the same good in other markets (e.g., Hausman (1996); Nevo (2001)), or shifts in the equilibrium concept (e.g., Porter (1983); Miller and Weinberg (2017)).[1] Both the covariance restrictions approach and the instruments approach rely on orthogonality conditions and therefore are conceptually linked. This connection is especially strong with the so-called "Hausman" instruments—prices of the same good in other markets—which are valid only if unobserved demand shocks are uncorrelated across markets. An important distinction, however, is that the instruments approach requires the econometrician to observe sufficient variation in some exogenous and excludable variable. The covariance restriction we implement, by contrast, does not require that *any* exogenous variation be observed. Rather, it allows

[1] See also Byrne et al. (2016) for an alternative set of instruments that leverages the structure of a standard model of discrete choice demand and differentiated-products price competition.

the econometrician to interpret the equilibrium variation in a manner consistent with the model.

Lastly, though we focus our results on specific, widely-used assumptions about demand and supply, we view our method as not particular to these assumptions. Rather, the main insight is that information about supply-side behavior can be modeled to adjust the observed relationships between quantity and price. Price can be decomposed into marginal cost and a markup; our method provides a direct way to correct for endogeneity arising from the latter component. In a more general sense, this insight has a similar flavor to control function estimation procedures (e.g., Heckman (1979)). Our method may be thought of a bias correction procedure for models commonly used in applications with imperfect competition.

2 A Motivating Example: Monopoly Pricing

We introduce the supply-side identification approach with a motivating example of monopoly pricing, in the spirit of Rosse (1970). In each market $t = 1, \ldots, T$, the monopolist faces a downward-sloping linear demand schedule, $q_t = \alpha + \beta p_t + \xi_t$, where q_t and p_t denote quantity and price, respectively, $\beta < 0$ is the price parameter, and ξ_t is mean-zero stochastic demand shock. Marginal cost is given by the function $mc_t = \gamma + \eta_t$, where γ is some constant and η_t is a mean-zero stochastic cost shock. Prices are set to maximize profit. The econometrician observes vectors of prices, $p = [p_1, p_2, \ldots, p_T]'$, and quantities, $q = [q_1, q_2, \ldots, q_T]'$. The markets can be conceptualized as geographically or temporally distinct.

It well known that an OLS regression of q on p obtains an estimate of β that is biased if the monopolist's price reflects the unobservable demand shock, as is the case here given the assumption of profit maximization. Formally,

$$\hat{\beta}^{OLS} = \frac{Cov(p,q)}{Var(p)} \quad \xrightarrow{p} \quad \beta + \frac{Cov(\xi,p)}{Var(p)} \tag{1}$$

The conventional wisdom is that $Cov(\xi, p) > 0$, such that OLS estimates understate the price elasticity of demand. Empirical studies often provides OLS estimates as a benchmark against which to evaluate IV estimates, but treat OLS as uninformative about the true parameter (e.g., Berry et al. (1995); Nevo (2001)). This approach can under-represent the informational content of OLS if the maintained supply-side assumptions inform $Cov(\xi, p) > 0$ and thus the magnitude of of OLS bias.

In our motivating example, the monopolist's profit-maximization conditions are such that price is equal to marginal cost plus a markup term: $p_t = \gamma + \eta_t - (\frac{dq}{dp})^{-1} q_t$. Thus, the numerator of the OLS bias can be decomposed into the covariance between demand shocks and markups and the covariance between demand shocks and marginal cost shocks. As we show in this paper, the former term may be consistently estimated. Therefore, when paired with additional knowledge about the covariance structure of shocks, we can consistently estimate and correct for the OLS bias, obtaining the true price parameter without the use of instruments.

In this paper, we will usually proceed under the assumption that the shocks to demand and marginal costs are uncorrelated. Uncorrelatedness may be a reasonable assumption for many settings, especially if panel data are used to absorb product-specific and time-specific factors (e.g., as in Nevo (2001)). Weaker assumptions, such as $Cov(\xi, \eta) \geq 0$, are insufficient for point identification but nonetheless allow the econometrician to place bounds on the price coefficient. We defer discussion along these lines until the following section. For now, we illustrate the identification approach using uncorrelatedness in the motivating example.

Proposition 1. *Let the OLS estimates of (α, β) be $(\hat{\alpha}^{OLS}, \hat{\beta}^{OLS})$ with probability limits $(\alpha^{OLS}, \beta^{OLS})$, and denote the residuals at the limiting values as $\xi_t^{OLS} = q_t - \alpha^{OLS} - \beta^{OLS} p_t$. When demand shocks and cost shocks are uncorrelated, the probability limit of the OLS estimate can be expressed as a function of the true price parameter, the residuals from the OLS regression, prices, and quantities:*

$$\beta^{OLS} \equiv plim\left(\hat{\beta}^{OLS}\right) = \beta - \frac{1}{\beta + \frac{Cov(p,q)}{Var(p)}} \frac{Cov(\xi^{OLS}, q)}{Var(p)} \tag{2}$$

Proof: We provide the proofs in this section for illustrative purposes; most subsequent proofs are confined to the appendix. Reformulate equation (1) as follows:

$$\beta^{OLS} = \beta + \frac{Cov(\xi, \eta - \frac{1}{\beta}q)}{Var(p)} = \beta - \frac{1}{\beta}\frac{Cov(\xi, q)}{Var(p)}$$

The first equality holds due to the first order condition $p = \gamma + \eta_t - \frac{1}{\beta}q$. The second equality holds due to the uncorrelatedness assumption. The structure of the model also allows for us to solve for $Cov(\xi, q)$:

$$\begin{aligned} Cov(\xi, q) &= Cov(\xi^{OLS} - (\beta - \beta^{OLS})p, q) \\ &= Cov(\xi^{OLS}, q) - (\beta - \beta^{OLS})Cov(p, q) \\ &= Cov(\xi^{OLS}, q) - \frac{1}{\beta}\frac{Cov(\xi, q)}{Var(p)}Cov(p, q) \end{aligned}$$

Collecting terms and rearranging implies

$$\frac{1}{\beta}Cov(\xi, q) = \frac{1}{\beta + \frac{Cov(p,q)}{Var(p)}} Cov(\xi^{OLS}, q)$$

Plugging into the reformulation of equation (1) obtains the proposition. QED.

The proposition makes clear that, among the objects that characterize the probability limit of the OLS estimate, only the true price parameter itself does not have a well understood sample analog. Because the probability limit itself can be estimated consistently, this raises the possibility that the true price parameter can be recovered from the data. Indeed, a closer inspection of equation (2) reveals that β solves a simple quadratic equation.

Proposition 2. *When demand shocks and marginal cost shocks are uncorrelated, β is point identified* *as the lower root of the quadratic equation*

$$\beta^2 + \beta\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right) + \left(-\frac{Cov(\xi^{OLS},q)}{Var(p)} - \frac{Cov(p,q)}{Var(p)}\beta^{OLS}\right) = 0 \tag{3}$$

and a consistent estimate of β is given by

$$\hat{\beta}^{\text{3-Step}} = -\sqrt{\left(\hat{\beta}^{OLS}\right)^2 + \frac{Cov(\hat{\xi}^{OLS},q)}{Var(p)}} \tag{4}$$

> **Proof:** The quadratic equation is obtained as a re-expression of equation (2). An application of the quadratic formula provides the following roots:
> $\frac{-\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right) \pm \sqrt{\left(\frac{Cov(p,q)}{Var(p)} - \beta^{OLS}\right)^2 + 4\left(\frac{Cov(\xi^{OLS},q)}{Var(p)} + \frac{Cov(p,q)}{Var(p)}\beta^{OLS}\right)}}{2}$.
> In the univariate case, $\frac{Cov(p,q)}{Var(p)} = \beta^{OLS}$, which cancels out terms and obtains the probability limit analog of equation (4). It is easily verified that $\left(\beta^{OLS}\right)^2 + \frac{Cov(\xi^{OLS},q)}{Var(p)} > 0$ so both roots are real numbers. The upper root is positive, so β is point identified as the lower root. The second equation of the proposition is the empirical analog to the lower root. As the sample estimates of covariance terms are consistent for the limits, it provides a consistent estimate of β.

The first part of the proposition states that β solves a quadratic equation. There are two real roots, but only one is negative, so point identification is achieved. Further, an adjustment to OLS estimator is sufficient to correct for bias. We label the estimator $\hat{\beta}^{\textit{3-Step}}$ for reasons that become evident with the more general treatment later in the paper. We recognize that this estimator is somewhat novel in the literature of industrial organization. Thus, to build confidence and intuition, we recast the monopoly problem in terms of supply and demand in Appendix A.1, and rederive the estimator building on Hayashi's (2000) textbook treatment of bias with simultaneous equations.

An important observation is that OLS residuals contain information sufficient to construct a consistent estimate of β. This is a result that generalizes to oligopoly settings and nonlinear demand systems, albeit with more difficult mathematics, as we develop in the subsequent sections. In our monopoly example, however, one additional simplification is available. Because $\xi_t^{OLS} = q_t - a^{OLS} - b^{OLS}p_t$, we have $\frac{Cov(\xi_t^{OLS},q)}{Var(p)} = \frac{Var(q)}{Var(p)} - \beta^{OLS}\frac{Cov(p,q)}{Var(p)}$. Plugging into equation (4) obtains the following corollary:

Corollary 1. *When demand shocks and marginal cost shocks are uncorrelated, a consistent estimate of β is given by* $\hat{\beta}^{\text{ALT}} = -\sqrt{\frac{Var(q)}{Var(p)}}$.

Thus, in the monopoly model, the price parameter can be estimated from the relative variation in equilibrium prices and quantities. Regression analysis is unnecessary.

We provide a simple numerical example to help fix ideas. Let demand be given by $q_t = 10 - p_t + \xi_t$ and let marginal cost be $mc_t = \eta_t$, so that $(\alpha, \beta, \gamma) = (10, -1, 0)$. Let the demand and cost shocks

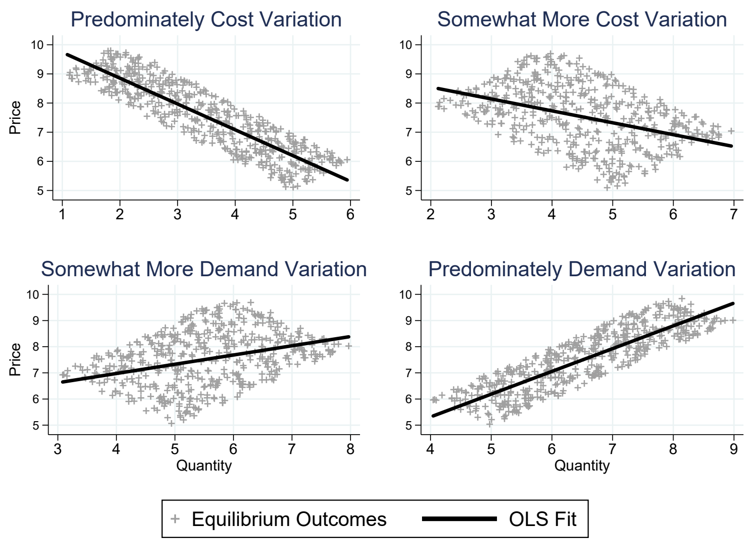

Figure 1: Price and Quantity in the Monopoly Model

have independent uniform distributions. The monopolist sets price to maximize profit. As is well known, if both cost and demand variation is present then equilibrium outcomes provide a "cloud" of data points that do not necessarily correspond to the demand curve. To illustrate this, we consider four cases with varying degrees of cost and demand variation. In case (1), $\xi \sim U(0,2)$ and $\eta \sim U(0,8)$. In case (2), $\xi \sim U(0,4)$ and $\eta \sim U(0,6)$. In case (3), $\xi \sim U(0,6)$ and $\eta \sim U(0,4)$. In case (4), $\xi \sim U(0,8)$ and $\eta \sim U(0,2)$. We randomly take 1,000 draws for each case and calculate the equilibrium prices and quantities.

The data are plotted in Figure 1 along with OLS fits. The experiment represents the classic identification problem of demand estimation: the empirical relationship between equilibrium prices and quantities can be quite misleading about the slope of the demand function. However, Proposition 2 and Corollary 1 state that the structure of the model together with the OLS estimates allow for consistent estimates of the price parameter. Table 1 provides the required empirical objects. The OLS estimates, $\hat{\beta}^{OLS}$, are negative when the cost shocks are relatively more important and positive when the demand shocks are relatively more important, as also is revealed in the scatterplots. By contrast, $\frac{Cov(\hat{\xi}^{OLS},q)}{Var(p)}$ is larger if the cost and demand shocks have relatively more similar support. Incorporating this adjustment term following Proposition 1 yields estimates, $\hat{\beta}^{\text{3-Step}}$, that are nearly equal to the population value of -1.00. The alternative estimator shown in Corollary 1, $\hat{\beta}^{ALT}$, is similarly accurate.

Table 1: Numerical Illustration for the Monopoly Model

	(1)	(2)	(3)	(4)
$\hat{\beta}^{OLS}$	-0.89	-0.42	0.36	0.88
$Var(q)$	1.47	1.11	1.08	1.38
$Var(p)$	1.45	1.09	1.06	1.37
$Cov(\hat{\xi}^{OLS}, q)$	0.31	0.92	0.94	0.32
$Cov(\hat{\xi}^{OLS}, q)/Var(p)$	0.21	0.85	0.89	0.24
$\hat{\beta}^{\textit{3-Step}}$	-1.004	-1.009	-1.009	-1.004
$\hat{\beta}^{ALT}$	-1.004	-1.009	-1.009	-1.004

Notes: Based on numerically generated data that conform to the motivating example of monopoly pricing. The demand curve is $q_t = 10 - p_t + \xi_t$ and marginal costs are $mc_t = \eta_t$, so that $(\beta_0, \beta, \gamma_0) = (10, -1, 0)$. In column (1), $\xi \sim U(0, 2)$ and $\eta \sim U(0, 8)$. In column (2), $\xi \sim U(0, 4)$ and $\eta \sim U(0, 6)$. In column (3), $\xi \sim U(0, 6)$ and $\eta \sim U(0, 4)$. In column (4), $\xi \sim U(0, 8)$ and $\eta \sim U(0, 2)$. Thus, the support of the cost shocks are largest (smallest) relative to the support of the demand shocks in the left-most (right-most) column.

3 Differentiated-Products Bertrand Competition

3.1 Data Generating Process

Let there be $j = 1, 2, \ldots, J$ firms that set prices in each of $t = 1, 2, \ldots, T$ markets, subject to downward-sloping demands. The econometrician observes vectors of prices, $p_t = [p_{1t}, p_{2t}, \ldots, p_{Jt}]'$, and quantities, $q = [q_{1t}, q_{2t}, \ldots, p_{Jt}]'$, corresponding to each market t, as well as a matrix of covariates $X_t = [x_{1t}\ x_{2t}\ \ldots\ x_{Jt}]$. The covariates are orthogonal to a pair of demand and marginal cost shocks (i.e.,. $E[X\xi] = E[X\eta] = 0$) [2] that are common knowledge among firms but unobserved by the econometrician. [3] We make the following assumptions on demand and supply:

Assumption 1 (Demand): *The demand schedule of each firm is determined by the following semi-linear form:*

$$h(q_{jt}; w_{jt}) = \beta p_{jt} + x'_{jt}\alpha + \xi_{jt} \tag{5}$$

where $h(q_{jt}; .)$ increases monotonically in its argument, i.e., $h'(q_{jt}; \cdot) > 0$, and where w_{jt} is a vector of observables and parameters that allow the semi-linear relationship.

The demand assumption restricts attention to systems for which, after a transformation of quantities, there is additive separability in prices, covariates, and the demand shock. The vector w_{jt} can be conceptualized as including the price and non-price characteristics of products, as well as demand parameters. With the discrete choice demand models that support research in industrial

[2] The presence of endogenous covariates requires supplemental moments to obtain identification. Our focus in this paper is on the estimation of the price parameter. It is straightforward to extend the estimator to incorporate additional restrictions.

[3] Another information environment consistent with this methodology is one in which each demand shock has a common component and an independent private component, and firms commit to prices before observing those of their rivals. In estimation, the common components may be captured by fixed effects, and then the rival firms' prices may then be included directly in X, as they will be orthogonal to the private demand shock for each firm.

organization, an empirical substitution for w_{jt} simplifies the transformation. For example, with the logit demand system, the market share of the outside good is a sufficient statistic for w_{jt}, such that $h(s_{jt}; w_{jt}) \equiv \log(s_{jt}/w_{jt})$, with $w_{jt} = s_{0t}$. With the more flexible random coefficients logit demand system, $h(q_{jt}; w_{jt})$ can be computed numerically via the contraction mapping of Berry et al. (1995). The demand assumption also nests monopolistic competition with linear demands (e.g., as in the motivating example). We derive these connections in some detail in Appendix B.

Assumption 2 (Supply): *Each firm sets price to maximize its profit in each market, taking as given the prices of other firms, with knowledge of the demand schedule equation* (5) *and the linear constant marginal cost schedule*

$$mc_{jt} = x'_{jt}\gamma + \eta_{jt}. \tag{6}$$

The supply assumption restricts attention to the constant marginal cost schedules that are commonly used in empirical industrial organization (e.g., Berry et al. (1995); Nevo (2001); Miller and Weinberg (2017)). Non-constant marginal costs can be incorporated, as we show in Section 4.1, but general solutions to endogenous regressors in cost function estimation lay outside the scope of this paper. Note that supply and demand may depend on different covariates; this is captured when non-identical components of α and γ are equal to zero. We maintain the assumption of Bertrand competition among single-product firms largely for mathematical convenience. In the empirical examples of Section 5 we extend results to Cournot competition and multi-product firms.

The first order conditions that must hold in equilibrium for each product j can be expressed:

$$p_{jt} = mc_{jt} - \frac{1}{\beta} h'(q_{jt}; w_{jt}) q_{jt}. \tag{7}$$

Prices are additively separable in marginal cost and a markup term. With constant marginal costs, prices respond to the unobserved demand shock solely through markup adjustments, which are fully determined by the price parameter, the structure of the model, and any nonlinear parameters in w_{jt}. This provides a basis for identification and estimation that we detail below. Lastly, we assume throughout that markets are "in equilibrium," in that the sense that prices are linked to an optimality condition that can be modeled by the researcher. In principle, the presence of multiple equilibria does not alter our results as long as the selected equilibrium can be identified by the econometrician.

3.2 Identification

We now formalize the identification argument for β, the price parameter. The model could feature parameters in w_{jt} that are tied to the nonlinear transformation or endogenous regressors, but these are not our focus and we assume they are known to the econometrician.[4] The linear non-price parameters (α, γ) can be recovered trivially if β and w_{jt} are known. We start by characterizing

[4] An alternative interpretation is that the econometrician is considering a candidate vector of nonlinear parameters, and wishes to determine the values that the linear parameters must take to rationalize the data. This alternative interpretation would apply in the nested fixed-point estimation routine of Berry et al. (1995) and Nevo (2001) for the random coefficients logit demand system.

the OLS estimate of the price parameter, which is obtained from a regression of $h(q_{jt};\cdot)$ on p. The probability limit of the estimator contains the standard bias term:

$$\beta^{OLS} \equiv \frac{Cov(p^*, h(q))}{Var(p^*)} = \beta + \frac{Cov(p^*, \xi)}{Var(p^*)} \tag{8}$$

where $p^* = [I - x(x'x)^{-1}x']p$ is a vector of residuals from a regression of p on x. Our first general result states that the probability limit of the OLS estimate depends on the true price parameter, the covariance between unobservables, and objects that can be directly estimated from the data.

Proposition 3. *Under assumptions 1 and 2, the probability limit of the OLS estimate can be written as a function of the true price parameter, the residuals from the OLS regression, the covariance between demand and supply shocks, prices, and quantities:*

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov(p^*, h'(q)q)}{Var(p^*)}} \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)} + \beta \frac{1}{\beta + \frac{Cov(p^*, h'(q)q)}{Var(p^*)}} \frac{Cov(\xi, \eta)}{Var(p^*)}. \tag{9}$$

The price parameter β solves the following quadratic equation:

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \beta^{OLS}\right)\beta \\ &+ \left(-\beta^{OLS}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)}\right). \end{aligned} \tag{10}$$

Proof. See appendix.

Proposition 3 provides our core identification result. There are two main implications. First, the quadratic in equation (10) admits (at most) two solutions for a given value of $Cov(\xi, \eta)$. It follows immediately that, with prior knowledge of $Cov(\xi, \eta)$, the price parameter β is set identified with a maximum of two elements (points). Indeed, as we show below, conditions exist that guarantee point identification. Second, if the econometrician does not have specific knowledge of $Cov(\xi, \eta)$, it nonetheless can be possible to bound β. We consider point identification first, as the intuition behind point identification maps neatly into how to construct bounds.

Assumption 3': *The econometrician has prior knowledge of $Cov(\xi, \eta)$.*

Proposition 4. ***(Point Identification)*** *For any value of $Cov(\xi, \eta)$, the price parameter β is the lower root of equation* (10) *if the following condition holds:*

$$0 \leq \beta^{OLS} \frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)} \tag{11}$$

and, furthermore, β *is the lower root of equation* (10) *if and only if the following condition holds:*

$$-\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*,-\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\eta\right)}{Var(p^*)} \tag{12}$$

Therefore, under assumptions 1, 2 and 3', β is point identified if either of these conditions holds.

Proof. See appendix.

The sufficient condition is derived as a simple application of the quadratic formula: if the constant term in the quadratic of equation (10) is negative then the upper root of the quadratic is positive and β must be the lower root. The condition can be evaluated empirically using only the data and assumptions 1 and 2. For some model specifications, it can be shown analytically from the structure of the model alone.[5] If the sufficient condition holds, then it is clear that β is point identified with prior knowledge of $Cov(\xi,\eta)$ because all the terms in equation (10) are known or can be obtained from the data. If the condition fails, point identification of β is not guaranteed even with prior knowledge of $Cov(\xi,\eta)$, and the econometrician has reduced the identified set to two points.

The necessary and sufficient condition is somewhat more nuanced. Even with prior knowledge of $Cov(\xi,\eta)$, condition (12) contains elements that are not observed by the econometrician. Still, in some specifications, the condition can be verified confirmed analytically.[6] When an analytical verification is not possible, the condition holds under the standard intuition that prices tend to increase both with demand shocks and marginal cost shocks, provided that $Cov(\xi,\eta)$ is not too positive. To see this in the equation, note that the term $-\frac{1}{\beta}\xi$ is the shock to the inverse demand curve. By contrast, the condition can fail if the empirical variation is driven predominately by demand shocks and the model dictates that prices decrease in the demand shock. This is a possibility if demand is log-convex (e.g., Fabinger and Weyl (2014)). If neither condition can be confirmed, then prior knowledge of $Cov(\xi,\eta)$ reduces the space of candidate values of β to the two roots of the quadratic in equation (10).

Without exact knowledge of $Cov(\xi,\eta)$, equation (10) can be used to construct bounds on β. The model implies two possible sets of (complementary) bounds. First, suppose the econometrician has a prior over the covariance between demand and cost shocks. For example, this can exist as a *plausible range*, such as $Cov(\xi,\eta) > 0$.[7] Using the mapping from each value of $Cov(\xi,\eta)$ to the (one

[5]An example is olipogoly with a linear demand system. Then we have that

$$\begin{aligned}\beta^{OLS}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)} &= \beta^{OLS}\frac{Cov(p^*,q)}{Var(p^*)} + \frac{Cov\left(q-\beta^{OLS}p^*,q\right)}{Var(p^*)} \\ &= (\beta^{OLS})^2 + \frac{Var(q)}{Var(p)} - \beta^{OLS}\frac{Cov(p^*,q)}{Var(p^*)} = \frac{Var(q)}{Var(p)} > 0\end{aligned}$$

[6]Consider again the example of an oligopoly facing a linear demand system, with the assumption $Cov(\xi,\eta)=0$. In the proof of Proposition 4, we show that the necessary and sufficient condition is equivalent to $\beta < \frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,h'(q)q)}{Var(p^*)}$. With linear demand, we have that $\frac{Cov(p^*,h'(q)q)}{Var(p^*)} = \frac{Cov(p^*,q)}{Var(p^*)} = \beta^{OLS}$. Thus, the right-hand-side simplifies to $-\beta$ using equation (8). Because $\beta<0$, $\beta<-\beta$ and the necessary and sufficient condition holds.

[7]In many cases, it might be natural to sign the correlation, especially if demand and supply are linked to period-specific

or two) values for β, we can construct a posterior set for the price parameter. When equation (11) holds, then the one-to-one mapping between the covariance and the price parameter will generate a convex set. Recall that condition (11) does not depend on $Cov(\xi, \eta)$. Likewise, one could verify that condition (12) holds for the plausible range of the covariance values to obtain bounds over a convex set for β.

The second set of bounds arises simply from the fact that the observed data and the data-generating process implied by the model must be compatible. Even when the econometrician has no prior about the covariance among unobservables, certain values of $Cov(\xi, \eta)$ can be ruled out by the data and the model structure. A necessary and sufficient condition for these bounds to bind is that both roots of equation (10) are negative, which can be verified empirically. Formally,

Proposition 5. ***(Model-Implied Bounds)*** *Under assumptions 1 and 2, if the sufficient condition for point identification fails, i.e.,*

$$0 > \beta^{OLS} \frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)}$$

then it must be that $Cov(\xi, \eta) \notin (c_1, c_2)$ *where* c_1 *and* c_2 *are defined by*

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Var(p^*)\beta^{OLS} - Cov(p^*, h'(q)q) - 2\sqrt{-Var(p^*)\left(Cov(p^*, h'(q)q)\beta^{OLS} + Cov(\xi^{OLS}, h'(q)q)\right)} \\ Var(p^*)\beta^{OLS} - Cov(p^*, h'(q)q) + 2\sqrt{-Var(p^*)\left(Cov(p^*, h'(q)q)\beta^{OLS} + Cov(\xi^{OLS}, h'(q)q)\right)} \end{bmatrix}$$

Proof. See appendix.

These bounds on $Cov(\xi, \eta)$ are "exclusive" in the sense that values of $Cov(\xi, \eta)$ that fall between the b_1 and b_2 can be ruled out. Evaluating the quadratic roots at these boundary values allows for bounds to be placed on β. Interestingly, these boundary values are available only if condition (11) fails. Thus, the econometrician may be unable to provide bounds while showing that point identification conditional on $Cov(\xi, \eta)$ is possible.

For illustrations of how to implement these bounds, see the empirical exercises in Section 5.

3.3 Estimation

For the purposes of estimation in this paper, we proceed under the assumption that unobserved demand shocks and marginal cost shocks are uncorrelated. We discuss empirical settings where this may be reasonable in Section 4.3. We formalize the assumption here:

Assumption 3 (Uncorrelatedness): $Cov(\xi, \eta) = 0$.

There are two natural approaches to estimation. The first is to apply the quadratic formula directly to equation (10). The second is to recast uncorrelatedness as a moment restriction of the form

macroeconomic shocks. To address this and achieve point identification, we suggest the use of fixed effects to account for this when the data allow for it.

$E[\xi'\eta] = 0$ and use the method of moments. Of these, the first is more novel, and so we open this section with the relevant theoretical result:

Corollary 2. *Under assumptions 1, 2, and 3, a consistent estimate of the price parameter β is given by*

$$\hat{\beta}^{\text{3-Step}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\hat{\xi}^{OLS}, h'(q)q\right)}{Var(p^*)}}\right)$$

if either condition (11) *or condition* (12) *holds.*

> **Proof:** Given uncorrelatedness, it can be shown that the right-hand-side of the equation is the empirical analog of the lower root of equation (10), and equals β under either of the two conditions. The empirical analog is consistent because sample estimates of the variance and covariance terms are consistent for their limits. QED.

We label the estimator $\hat{\beta}^{\textit{3-Step}}$ because there are three steps in the estimation procedure: (i) regress $h(q)$ on p and x with OLS, (ii) regress p on x with OLS and construct the residuals p^*, and (iii) construct the estimator as shown.[8] If condition (11) does not hold and the model does not allow for an analytical evaluation of condition (12), then $\hat{\beta}^{\textit{3-Step}}$ remains consistent for the lower root of equation (10), but this is not guaranteed to be β. In that case, both roots can be estimated and evaluated as candidate parameters. Even though both are negative, the upper root still may be possible to rule out if it requires implausible marginal costs to rationalize observed prices.

The uncorrelatedness assumption imposes the moment condition $E[\xi'\eta] = 0$. Given the maintained assumptions $E[X\xi] = 0$ and $E[\xi] = 0$ and the structure of marginal costs in assumption 2, these moments can be combined into the (stronger) condition $E[\xi' mc] = 0$. An alternative approach is estimation is to search numerically to find a price parameter, $\hat{\beta}$, that satisfies the empirical moment

$$\frac{1}{T}\sum_t \frac{1}{|J_t|}\sum_{j\in J_t} \xi_{jt}(\hat{\beta}; w, X)\cdot mc_{jt}(\hat{\beta}; w, X) = 0$$

where $\xi(\hat{\beta}; w, X)$ and $mc(\hat{\beta}; w, X)$ are computed given the data and the price parameter using equations (5)-(7). The firms present in each market t are indexed by the set J_t. Formally, this method-of-moments estimator is defined as

$$\hat{\beta}^{MM} = \arg\min_{\hat{\beta}<0}\left[\frac{1}{T}\sum_t \frac{1}{|J_t|}\sum_{j\in J_t} \xi_{jt}(\hat{\beta}; w, X)\cdot mc_{jt}(\hat{\beta}; w, X) = 0\right]^2$$

The linear parameters (α, γ) are concentrated out of the nonlinear optimization problem. The conditions for identification are identical to those of the three-step estimator. Indeed, in numerical experiments we have confirmed that the two estimators are equivalent to numerical precision, taking care to ensure that method-of-moments converges to the lower root of equation (10).

[8] A more precise two-step estimator is available for special cases in which the observed cost and demand shifters are uncorrelated. See Appendix C for details.

Comparing the two approaches, it is clear that the three-step estimator has distinct advantages over the method-of-moments approach. The analytic formula reduces the computational burden in estimation. This can be especially beneficial when nested inside of a nonlinear routine for other parameters because multi-dimensional optimization can be particularly burdensome. Additionally, the three-step estimator reduces computational error by immediately obtaining the correct solution and rejecting cases when no solution is possible. A method-of-moments optimizer may converge at a local minimum, and it produces a solution even when the moment may not be set to zero (which should reject the model).

On the other hand, the numerical approach to estimation also offers some advantages. First, there are settings for which the three-step estimator does not generalize or is difficult to calculate. Examples include models in which demand is not semi-linear or the first-order conditions for multi-product firms are complicated (though see Section 5.2). The method-of-moments estimator handles such settings without difficulty. Second, the empirical moments can be combined with more standard exclusion restrictions to construct a generalized method-of-moments estimator. This can improve efficiency and allows for specification tests that otherwise would be unavailable to the econometrician (e.g., Hausman (1978); Hansen (1982)). Finally, the three-step estimator requires orthogonality between the unobserved demand shock and *all* the regressors. The method-of-moments approach can be pursued under a weaker assumption that allows for correlation between the unobserved demand shock and regressors that enter the cost function only. In this case, one would replace $E[\xi' mc] = 0$ with $E[\xi'\eta] = 0$ in the objective function.[9]

3.4 Small-Sample Properties

We generate Monte Carlo results to examine the small sample properties of the estimators. We consider a profit-maximizing monopolist that prices against a logit demand curve and has a constant marginal cost technology:

$$\begin{aligned} h(q_t; w_t) &\equiv \log(q_t) - \log(1 - q_t) = -\beta p_t + \xi_t \\ mc_t &= x_t + \eta_t \end{aligned}$$

For simplicity, we set $\beta = 1$ and simulate data for x, ξ, η using independent $U[0, 1]$ distributions. For each draw of the data, we compute profit-maximizing prices and quantities. The mean price and margin are 2.20 and 0.56, respectively, and the mean price elasticity of demand is -1.86. We construct samples with 25, 50, 100, and 500 observations and estimate demand with each. We repeat this exercise 1,000 times and examine the average and standard deviation of the estimates. The estimators are the 3-Step estimator, 2SLS using x_t as an instrument, a method-of-moments ("MM") estimator based on the alternative moment $E[\xi'\eta] = 0$, and OLS.

Table 2 summarizes the results. The bias present in 3-Step, 2SLS, and MM is small even with the smallest sample sizes. However, 3-Step more consistently provides accurate estimates than 2SLS

[9] Finite-sample numerical experiments suggest that estimation based on $E[\xi'\eta] = 0$ tends to produce greater bias and mean squared error than estimation based on $E[\xi' mc] = 0$, if both assumptions are valid. This reflects that the latter moment incorporates the additional restriction $mc_t = X_t\gamma + \eta_t$.

Table 2: Small Sample Properties of Estimators

Panel A: Average Estimates (Truth is $\beta = -1.00$)				
Sample Size	3-Step	2SLS	MM	OLS
25	-1.002	-1.008	-1.005	-0.885
50	-1.004	-1.012	-1.002	-0.889
100	-1.004	-1.006	-1.005	-0.891
500	-1.000	-1.001	-0.999	-0.887

Panel B: Standard Deviation of Estimates				
Sample Size	3-Step	2SLS	MM	OLS
25	0.160	0.276	0.208	0.168
50	0.109	0.182	0.141	0.114
100	0.078	0.123	0.101	0.082
500	0.035	0.053	0.045	0.037

Notes: The moments used for 3-Step, 2SLS, MM, and OLS are $E[\xi' mc]$, $E[\xi' x]$, $E[\xi' \eta]$, and $E[\xi' y]$, respectively. The methods-of-moments ("MM") estimator is implemented with a one-dimensional grid search.

and MM, as evidenced by the smaller standard deviation of the estimates. The reason is that 3-Step utilizes orthogonality between unobserved demand and marginal cost, whereas 2SLS and MM exploit the relationship between unobserved demand and *marginal cost shifters*—either observed (x_t) or unobserved (η_t)—which provide noisy signals about marginal cost. One might be tempted to run a "first-stage" regression to test for the power of the different cost components to predict prices. However, such a test has no bearing on the asymptotic properties of the 3-Step and MM estimators. This is an important conceptual observation: with 3-Step and MM, exogenous supply-side variation *need not be observed by the econometrician* and indeed *need not even exist*. Recall the monopoly experiments of Section 2, in which the price parameter was be recovered from the data even if all the variation in data arise from the demand-side. This is both a strength and a weakness: relaxing the requirement of observed exogenous variation comes at the cost of a greater reliance on assumptions about how firms set prices in equilibrium.

4 Discussion and Robustness

Our estimation approaches developed above rely on an accurate model of the data-generating process and some relatively strong (though somewhat widespread) restrictions on the form of demand and supply. Some of these assumptions can be relaxed with relative ease, and we explore particular extensions–such as Cournot competition and multi-product firms–in our empirical analysis. Here, we consider three assumptions worthy of discussion in more detail: constant marginal costs, specific forms of competition, and uncorrelatedness itself.

4.1 Non-Constant Marginal Costs

If marginal costs are not constant in output, then unobserved demand shocks affect price both through markup adjustments and via their impacts on marginal cost. For example, consider a special case in which marginal costs take the form:

$$mc_{jt} = x'_{jt}\gamma + g(q_{jt}; \lambda) + \eta_{jt} \tag{13}$$

Here $g(q_{jt}; \lambda)$ is some potentially nonlinear function that may (or may not) be known to the econometrician. Maintaining Bertrand competition and the baseline demand assumption, the first order conditions of the firm are:

$$p_{jt} = \underbrace{x'_{jt}\gamma + g(q_{jt}; \lambda) + \eta_{jt}}_{\text{Marginal Cost}} + \underbrace{\left(-\frac{1}{\beta}h'(q_{jt}; w_{jt})q_{jt}\right)}_{\text{Markup}}.$$

The OLS regression of $h(q_{jt}; w_{jt})$ on price and covariates now yields a price coefficient with the following probability limit:

$$plim(\hat{\beta}^{OLS}) = \beta - \frac{1}{\beta}\frac{Cov(\xi, h'(q)q)}{Var(p^*)} + \frac{Cov(\xi, g(q))}{Var(p^*)}$$

Provided $g'(\cdot; \lambda) \neq 0$, it no longer is the case that the relationship between prices and unobserved demand shocks are fully determined by the price parameter, the structure of the model, and any nonlinear parameters in w_{jt}. This precludes identification of β using the results of the preceding section, unless some knowledge of $g(q_{jt}; \lambda)$ can be brought to bear on the problem.

There are two ways to make progress. First, if $g'(\cdot; \lambda)$ can be signed then it is possible to *bound* the price parameter, β, even if point identification remains infeasible. A lead example is that of capacity constraints, for which it might be reasonable to assume that $Cov(\xi, \eta) = 0$ and $g'(\cdot; \lambda) \geq 0$, and thus that $Cov(\xi, \eta^*) \geq 0$ where $\eta^*_{jt} = \eta_{jt} + g(q_{jt}; \lambda)$ is a composite error term. Applying $Cov(\xi, \eta^*) \geq 0$ to the lower root of the quadratic in equation (10) then can tighten the set of candidate price parameters. Second, the econometrician may be able to estimate $g(q_{jt}; \lambda)$, either in advance of or simultaneously with the price coefficient. Our main theoretical result of the section is that prior knowledge of $Cov(\xi, \eta)$ is sufficient to at least set identify β in such a situation.

Proposition 6. *Under assumptions 1 and 3 and a modified assumption 2 such marginal costs take the semi-linear form of equation* (13)*, the price parameter β solves the following quadratic equation:*

$$\begin{aligned}
0 \quad = \quad & \left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\beta^2 \\
+ \quad & \left(\frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, g(q))}{Var(p^*)}\right)\beta \\
+ \quad & \left(-\frac{Cov(p^*, h'(q)q)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\hat{\xi}^{OLS}, h'(q)q)}{Var(p^*)}\right)
\end{aligned}$$

where $\hat{\beta}^{OLS}$ is the OLS estimate and $\hat{\xi}^{OLS}$ is a vector containing the OLS residuals.

Proof. See appendix.

With the above quadratic in hand, the remaining results of Section 3 extend naturally. Although the estimation of $g(q_{jt};\lambda)$ is not our focus, we note that a 3-Step estimator of β could be obtained for any candidate parameters in λ, thereby facilitating computational efficiency.

4.2 Various Forms of Competition

Though our main results are presented using Bertrand-Nash competition in prices, our method is robust to any form of competition where price can be expressed as the marginal cost plus a markup term. This general form for equilibrium prices, $p = c + \mu$, is obtained under a broad set of assumptions. Consider, for example, Nash competition among profit-maximizing firms that have a single choice variable, a, and constant marginal costs. The individual firm's objective function is:

$$\max_{a_j|a_i, i\neq j} (p_j(a) - c_j)q_j(a).$$

The first-order condition, holding fixed the actions of the other firms, is given by:

$$p_j(a) = c_j - \frac{p_j{}'(a)}{q_j{}'(a)} q_j(a).$$

Thus, in a generalized form that nests Betrand ($a = p$) and Cournot ($a = q$), prices are of the form $p = c + \mu$, where the μ incorporates the structure of demand and key parameters. Importantly, marginal costs and the markup are additively separable. Additively separability in this context provides a helpful restriction on how prices move with demand shocks, aiding identification. The bias from OLS estimates can be decomposed into additively separable components consisting of how (residualized) prices covary with marginal costs and how prices covary with the markup.

For any competition specification that results in the additively separable components, it is straightforward to extend our three-step results, as the semi-linear demand schedule will produce a $1/\beta$ term in the markup. In addition to the single-action Nash form considered above, additive separability (and our three-step method) translate to competition in quantities with increasing marginal costs, multi-product firms, and certain forms of non-Nash competition. We explore the first two extensions in Section 5.

4.3 Uncorrelatedness

Finally, we turn to the uncorrelatedness assumption itself. There are a variety of reasons that marginal costs may be correlated with demand unobservables. For instance, unobserved quality is more costly to supply and also increases demand. Additionally, firms may be induced to produce further along an upward-sloping marginal cost schedule if demand is stronger. In many cases, however, econometric adjustments such as the incorporation of fixed effects may capture the primary factors that drive this correlation, rendering the uncorrelatedness assumption more palatable.

To illustrate, consider the following generalized demand and cost functions:

$$\begin{aligned} h(q_{jt}; w_{jt}) &= \beta p_{jt} + x'_{jt}\alpha + D_j + F_t + E_{jt} \\ mc_{jt} &= g(q_{jt}, v_{jt}; \lambda) + x'_{jt}\gamma + U_j + V_t + W_{jt} \end{aligned}$$

Let the unobserved demand shocks be $\xi_{jt} = D_j + F_t + E_{jt}$ and the unobserved costs be $\eta_{jt} = U_j + V_t + W_{jt}$. Further assume that the $h(\cdot)$ and $g(\cdot)$ functions are known (up to parameters). If products with higher quality have higher marginal costs then $Cov(U_j, D_j) > 0$. With panel data, the econometrician can account for the relationship by estimating D_j for each firm. The residual $\xi^*_{jt} = \xi_{jt} - D_j$ is uncorrelated with U_j so that uncorrelatedness applies. Similarly, if costs are higher (or lower) in markets with high demand then F_t can be estimated with panel data such that uncorrelatedness applies.

Thus, if panel data permit the inclusion of product and market fixed effects then the remaining unobserved correlation, $Cov(W_{jt}, E_{jt})$, involves firm-specific demand and cost deviations within a market, and the assumption of uncorrelatedness may become reasonable across a wide range of applications. This is not to argue that uncorrelatedness is universally valid. Even in the presence of product and market fixed effects, a number of mechanisms could create an empirical relationship between the unobserved demand and cost shocks. The assumption would be violated if some firms offer temporary per-unit incentives to sales representatives, or if market power in input markets creates a relationship between demand and input prices. If such mechanisms are left outside the model then the estimators derived above do not provide consistent estimates of demand. However, it still may be possible to sign the correlation between unobserved terms, in which case it may be possible to construct bounds on the demand parameters.

5 Empirical Examples

5.1 The Portland Cement Industry

Our first empirical example uses the setting and data of Fowlie et al. (2016) ["FRR"], which examines market power in the cement industry and its effects on the efficacy of environmental regulation. The model features Cournot competition among cement plants facing capacity constraints. Our theoretical results apply with minor modifications. We find that our three-step estimator with uncorrelatedness produces similar demand elasticities to 2SLS with instruments, so long as capacity constraints are accounted for on the supply-side. If, instead, the impact of capacity constraints on marginal costs is unknown *a priori* then it is possible to bound the demand elasticities.

We begin with the theoretical extension. Let $j = 1, \ldots, J$ firms produce a homogeneous product demanded by consumers according to $h(Q; w) = \beta p + x'\gamma + \xi$, where $Q = \sum_j q_j$, and p represents a price common to all firms in the market. Marginal costs are semi-linear, as in equation (13), possibly reflecting capacity constraints. Working with aggregated first order conditions, it is possible to show that the OLS regression of $h(Q; w_{jt})$ on price and covariates yields:

$$plim(\hat{\beta}^{OLS}) = \beta - \frac{1}{\beta}\frac{1}{J}\frac{Cov(\xi, h'(Q)Q)}{Var(p^*)} + \frac{Cov(\xi, \overline{g})}{Var(p^*)}$$

where J is the number of firms in the market and $\overline{g} = \frac{1}{J}\sum_{j=1}^{J} g(q_j; \lambda)$ is the average contribution of $g(q, \lambda)$ to marginal costs. Bias arises due to markup adjustments and the correlation between unobserved demand and marginal costs generated through $g(q; \lambda)$.[10] The identification result provided in 4.1 for models with non-constant marginal costs extends naturally.

Corollary 3. *In the Cournot model, the price parameter β solves the following quadratic equation:*

$$\begin{aligned} 0 &= \left(1 - \frac{Cov(p^*, \overline{g})}{Var(p^*)}\right)\beta^2 \\ &+ \left(\frac{1}{J}\frac{Cov(p^*, h'(Q)Q)}{Var(p^*)} + \frac{Cov(\xi, \overline{\eta})}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, \overline{g})}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\hat{\xi}^{OLS}, \overline{g})}{Var(p^*)}\right)\beta \\ &+ \left(-\frac{1}{J}\frac{Cov(p^*, h'(Q)Q)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{1}{J}\frac{Cov(\hat{\xi}^{OLS}, h'(Q)Q)}{Var(p^*)}\right) \end{aligned}$$

The derivation tracks exactly the proof of Proposition 6. For the purposes of empirical exercise, we compute the 3-Step estimator as the empirical analog to the lower root of this quadratic.

We turn now to the empirical setting. Let the demand curve in each market have a logit form:

$$h(Q_{rt}; w) \equiv \ln(Q_{rt}) - \ln(M_r - Q_{rt}) = \alpha_r + \beta p_{rt} + \xi_{rt}$$

where r and t denote region and time, respectively, and M_r is the size of the region. We assume $M_r = 2 \times \max_t\{Q_{rt}\}$ for simplicity. Marginal costs take a "hockey stick" form:

$$\begin{aligned} mc_{jrt} &= \gamma + g(q_{jrt}) + \eta_{jrt} \\ g(q_{jrt}) &= 2\lambda_2 1\{q_{jrt}/k_{jr} > \lambda_1\}(q_{jrt}/k_{jr} - \lambda_1) \end{aligned}$$

where k_{jr} and q_{jrt}/k_{jr} are capacity and utilization, respectively. Marginal costs are constant if utilization is less then the threshold $\lambda_1 \in [0, 1]$, and increasing linearly at rate determined by $\lambda_2 \geq 0$ otherwise.[11] The assumption $Cov(\xi, \eta) = 0$ is reasonable because the model incorporates explicitly how capacity constraints affect marginal cost.

Table 3 summarizes the results of demand estimation. The 3-Step estimator is implemented taking as given the nonlinear cost parameters obtained in FRR: $\lambda_1 = 0.869$ and $\lambda_2 = 803.65$. In principle, these could be estimated simultaneously via the method of moments, provided some demand shifters can be excluded from marginal costs, but that is not our focus. As shown, the mean

[10] Bias due to markup adjustments dissipates as the number of firms grows large. Thus, if marginal costs are constant then the OLS estimate is likely to be close to the population parameter in competitive markets. While this explicit analytical result is particular to the Cournot model, we find that a similar effect arises in Monte Carlo experiments based on Bertrand competition and logit demand.

[11] We specify a logit demand curve rather than the constant elasticity demand curve of FRR because it more easily conforms to our framework. The 2SLS results are unaffected by the choice. Similarly, the 3-Step estimator with logit obtains virtually identical results as a method-of-moments estimator with constant elasticity demand that imposes uncorrelatedness.

Table 3: Point Estimates for Cement with Uncorrelatedness

Estimator:	3-Step	2SLS	OLS
Elasticity	-1.15	-1.07	-0.47
of Demand	(0.18)	(0.19)	(0.14)

Notes: The sample includes 520 region-year observations over 1984-2009. Bootstrapped standard errors are based on 200 random samples constructed by drawing regions with replacement.

price elasticity of demand obtained with the 3-Step estimator under uncorrelatedness is -1.15. This is statistically indistinguishable from the 2SLS elasticity estimate of -1.07, which is obtained using the FRR instruments: coal prices, natural gas prices, electricity prices, and wage rates. Both 3-Step and 2SLS move the elasticities in the expected direction relative to OLS.[12]

If the econometrician does not know (and cannot identify) the nonlinear parameters in the cost function, then consistent estimates cannot be obtained with our methodology. Some progress can be made nonetheless. Defining the composite marginal cost shock, $\eta^*_{jrt} = g(q_{jrt}) + \eta_{jrt}$, we have $Cov(\xi, \overline{\eta}^*) \geq 0$ if $Cov(\xi, \overline{\eta}) = 0$. This is sufficient to bound on the demand elasticity below -0.69. The OLS estimate thus can be ruled out, by a fair margin.[13] Finally, we examine the sensitivity of the point estimates to the uncorrelatedness assumption. Table 4 provides the correlation coefficients (i.e., $Cor(\xi^{alt}, \eta)$) and elasticities obtained for different levels of $Cov(\xi^{alt}, \eta)$. As the (imposed) covariance term becomes more negative, the price parameter converges to zero and the correlation coefficient converges to approximately -0.37. As the covariance becomes more positive, the price parameter converges to $-\infty$ and the correlation coefficient converges to approximately 0.56. If the researcher believes that the correlation between unobservables is small, say under 0.10 in magnitude, then the implied demand elasticities remain reasonably close to the point estimates.

5.2 The Airline Industry

Our second empirical exercise uses the setting and data of Aguirregabiria and Ho (2012) ["AH"], which explores why airlines form hub-and-spoke networks.[14] The model features differentiated-products Bertrand competition among multi-product firms facing a nested logit demand system. Our theoretical results extend to multi-product firms in this context with minor modifications. We show that some negative values of the price parameter, β, cannot be rationalized given the model and data if the nesting parameter, $\sigma \in [0, 1)$, is relatively large. We provide identifying conditions—building on uncorrelatedness—under which σ can be identified without instruments. Finally, we estimate the model with and without instruments, and compare results.

[12]Under uncorrelatedness and constant marginal costs ($\lambda_2 = 0$) the estimated mean price elasticity is -0.79, near the midpoint between the 3-Step and OLS estimates reported in Table 3. Both markup adjustments and non-constant marginal costs both contribute meaningfully to bias in OLS.

[13]We obtain this point by computing the lower root of the quadratic in Corollary 3 under the assumptions that $Cov(\xi, \overline{\eta}) = 0$ and $\lambda_1 = \lambda_2 = 0$. We confirm that this provides an upper bound by examining the lower root under the alternative assumption that $Cov(\xi, \overline{\eta})$ is positive.

[14]We thank Victor Aguirregabiria for providing the data. Replication is not exact because the sample differs somewhat from what is used in the AH publication and because we employ a different set of fixed effects in estimation.

Table 4: Sensitivity of Point Estimates for Cement

Correlation Interval	Elasticity Interval
(-0.02,0.02)	(-1.10, -1.20)
(-0.05,0.05)	(-1.03, -1.29)
(-0.10,0.10)	(-0.91, -1.44)
(-0.20,0.20)	(-0.70, -1.81)
(-0.30,0.30)	(-0.51, -2.34)
⋮	⋮
(-0.35,0.56)	(0.00, $-\infty$)

Notes: The model is estimated using a modified 3-Step estimator that imposes $Cov(\xi, \eta) = s$, where s takes discrete values in [-100,100]. For each s we calculate the correlation coefficient and elasticity. The correlation coefficients are bounded even though the covariance is not.

To provide context, we first formulate the nested logit demand system, following Berry (1994) and Cardell (1997). Inverting the market share equation yields

$$h(s_j; w) \equiv \ln s_j - \ln s_0 - \sigma \ln \overline{s}_{j|g} = \beta p_j + x_j' \alpha + \xi_j \tag{14}$$

where s_j is the market share and $\overline{s}_{j|g} = s_j / \sum_{k \in g} s_k$ is the *conditional market share*, i.e., the choice probability of product j given that its "group" of products, g, is selected. The outside good is indexed as $j = 0$. Higher values of σ increase within-group consumer substitution relative to across-group substitution. We now provide the theoretical extension to multi-product firms.

Assumption 4: *The derivatives of the transformation parameters with respect to prices, $\frac{dw_{kt}}{dp_{jt}}$, are linear in β.*

The assumption applies with the nested logit model because $(s_0, \overline{s}_{j|g}, \sigma)$ are sufficient statistics for w and it can be verified that $\frac{\partial s_0}{\partial p_j}$ and $\frac{\partial \overline{s}_{j|g}}{\partial p_j}$ are linear in β. Under assumption 4, the first order conditions extend tractably to the multi-product setting. Letting $\tilde{h}$ be the multi-product analog for $h'(q)q$, we obtain a quadratic in β, and the remaining results of Section 3 then obtain easily:

Proposition 7. *Under assumptions 1, 2, 3, and 4, the price parameter β solves the following quadratic equation:*

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*, \tilde{h})}{Var(p^*)} + \frac{Cov(\xi, \eta)}{Var(p^*)} - \hat{\beta}^{OLS} \right) \beta \\ &+ \left(-\frac{Cov(p^*, \tilde{h})}{Var(p^*)} \hat{\beta}^{OLS} - \frac{Cov(\hat{\xi}^{OLS}, \tilde{h})}{Var(p^*)} \right) \end{aligned}$$

where $\tilde{h}$ is constructed from the first-order conditions of multi-product firms.

Proof. See appendix.

The data are drawn from the *Airline Origin and Destination Survey* (DB1B) survey, a ten percent sample of airline itineraries, for the four quarters of 2004. Markets are directional round trips between origin and destination cities in a particular quarter. Products are nonstop or one-stop itineraries from a particular airline. Thus, each airline can have zero, one, or two products per market. The nesting parameter, σ, governs consumer substitution across three product groups: nonstop flights, one-stop flights, and the outside good. Marginal costs are linear in accordance with equation (6). Following AH, the covariates include an indicator for nonstop itineraries, the distance between the origin and destination cities, and a measure of the airline's "hub sizes" at the origin and destination cities. We also include airline fixed effects and route×quarter fixed effects. The latter expands on the city×quarter fixed effect used in AH. Market size, which determines the market share of the outside good, is equal to the total population in the origin and destination cities.

We start with an analysis of model-implied bounds, based on Proposition 5, which does not require assumptions on $Cov(\xi, \eta)$. Panel A of Figure 2 shows that some intermediate values of $Cov(\xi, \eta)$ can be rejected if $\sigma \geq 0.62$. Uncorrelatedness is rejected with $\sigma \geq 0.69$ and, as $\sigma \to 1$, it must be that $Cov(\xi, \eta) \leq -0.64$ or $Cov(\xi, \eta) \geq 0.35$. Panel B provides the corresponding bounds on β. As σ becomes larger, a more negative β is required to rationalize the data within the context of the model. If $\sigma = 0.80$ then it must be that $\beta \leq -0.11$. The panel also plots the 3-Step estimator, $\hat{\beta}^{3-Step}(\sigma)$, over its supported range. Unlike the bounds, the estimator assumes that $Cov(\xi, \eta) = 0$. As σ converges to 0.69 from below, the 3-Step estimator approaches both the bound on β and the upper root of the Proposition 7 quadratic. The upper root can be ruled out as an estimate of β even when it is negative because it violates the model-implied bounds.[15]

In the nested logit model, conditional shares respond to the unobserved demand shock, which creates a second endogenous variable in addition to price. Thus, uncorrelatedness must be supplemented with an additional moment if both β and σ are to be estimated. Intuitively, the three-step estimator is consistent for β *conditional* on transformation parameters (i.e., σ), but it does not recover the transformation parameters themselves. If instruments are available, estimation can be done with the method-of-moments, pairing the orthogonality conditions $E[\xi'\eta] = 0$ and $E[Z\xi] = 0$. The econometrician could search numerically over the parameters spaces of β and σ simultaneously, minimizing a weighted sum of the moments (squared). Alternatively, the econometrician could conduct a single-dimensional search, obtaining $\hat{\beta}^{3-Step}(\tilde{\sigma})$ for each candidate transformation parameter $\tilde{\sigma}$ while minimizing the supplemental modments $E[Z\xi] = 0$. We implement the latter approach, the presence of two exogenous variables that appear in the AH marginal cost function but are excluded from the AH demand function.[16] These supplemental moments differ from the AH instruments included in 2SLS estimation, which we describe below.

An alternative path to the joint estimation of β and σ is available if the notion of uncorrelatedness can be extended to grouped moments or higher-order moments. This approach does not require

[15] It possible that the upper root can *always* be ruled out on this basis. We are working to generalize the result.

[16] In the demand equation, hub size of any given city-airline pair is the sum of population in other cities that the airline connects with direct itineraries from the city. In the supply equation, this is replaced with an analogous measure based on the number of connections rather than population.

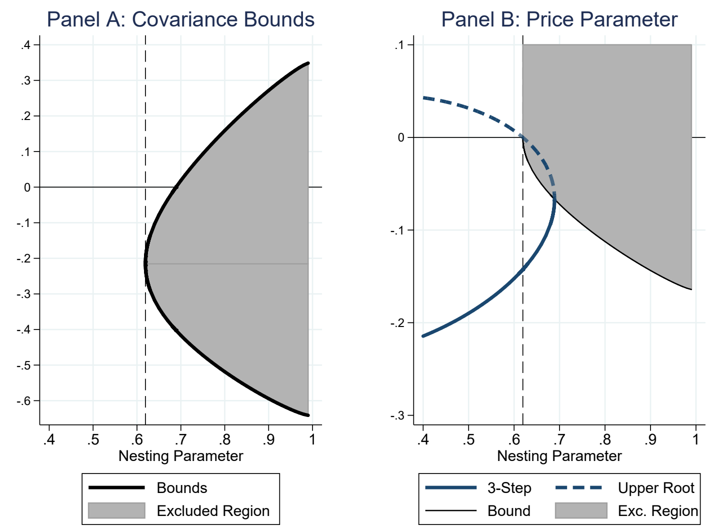

Figure 2: Model-Implied Bounds for Airlines

Table 5: Application to U.S. Airlines

Parameter	3-Step-I	3-Step-II	3-Step-III	2SLS	OLS
β	-0.182	-0.153	-0.131	-0.189	-0.106
	(0.042)	(0.031)	(0.031)	(0.053)	(0.004)
σ	0.525	0.599	0.639	0.822	0.891
	(0.110)	(0.113)	(0.140)	(0.087)	(0.003)

Notes: The first two columns of results use the three-step methodology with different supplemental moments. The next column use 2SLS and OLS, respectively, following Aguirregabiria and Ho (2012). Standard errors are constructed via subsamples of 100 market-periods. There are 93,199 observations and 11,474 market-periods in the full sample.

the econometrician to be able to isolate exogenous variation in prices and conditional shares. We implement with two supplementary assumptions:

- $Cov(\overline{\xi}, \overline{\eta}) = 0$ where $\overline{\xi}_{gt} = \frac{1}{|g|}\sum_{j\in g}\xi_{jt}$ and $\overline{\eta}_{gt} = \frac{1}{|g|}\sum_{j\in g}\eta_{jt}$ are the mean demand and cost shocks within a group-market pair. This is a simple refinement of uncorrelatedness: the mean shocks within a product group are uncorrelated across groups and markets.

- $Cov(\xi^2, \eta) = 0$ and $Cov(\xi, \eta^2) = 0$. These identifying assumptions state that the variance of one shock is uncorrelated with the level of the other shock.

Note that the latter assumption does not provide independent identifying power if (ξ, η) are jointly normal distribution, because then it is implied by uncorrelatedness.

Table 5 summarizes the results of estimation. The left three columns are obtained with the three-step methodology sketched above, using first $E[Z\xi] = 0$ as a supplemental moment, then using $Cov(\overline{\xi}, \overline{\eta}) = 0$, and finally using $Cov(\xi^2, \eta) = 0$ and $Cov(\xi, \eta^2) = 0$. The fourth column is obtained with 2SLS using the AH instruments that are not absorbed by route×period fixed effects: the average hub-sizes (origin and destination) of all other airlines on the route and the average value of the nonstop indicator for all the other carriers on the route. The final column is obtained with OLS. The 3-Step estimators and 2SLS all move the parameters in the expected direction relative to OLS. A cross-check with the bounds analysis above confirms that each set of bias-corrected estimates can be rationalized within the model, but that the OLS estimates are inconsistent with the model for any covariance structure and are rejected. Comparing the estimates, 3-Step produces less negative price parameters and smaller nesting parameters than 2SLS. We do not seek to ascertain which set of estimates is more in line with real-world behavior.[17]

6 Empirical Application

Coming soon.

7 Conclusion

The objective of this paper is to develop and evaluate the conditions under which supply-side assumptions typically made in empirical models of imperfect competition can be used to address the econometric problem of price endogeneity. The main result is that a covariance restriction between the unobserved demand and marginal cost components often is sufficient to obtain point identification. The empirical exercises we offer demonstrate the practicality of this approach in settings where uncorrelatedness is a reasonable assumption. While the identifying power of such covariance restrictions has previously been demonstrated in a number of models, this has not been extended to models of imperfect competition, which feature the complication that structural parameters from one side of the model (demand) enter the other side of the model (supply) through markups. We speculate that this complication explains why covariance restrictions of the kind we propose are not currently part of the standard industrial organization "toolkit."

Our main result has a number of interesting implications. Chief among these is that identification does not require a source of exogenous variation if the econometrician has a sufficiently tight model of the data generating process. Because the literature of industrial organization has long couched the identification challenge in terms of exogenous variation and the search for instruments, we conclude with a short set of observations about the two approaches. First, the reasonableness of covariance restriction between unobservables should be evaluated in light of the institutional details of the application, receiving a similar level of scrutiny currently place on orthogonality assumptions between instruments and unobservables. Second, for some applications, valid instruments are unavailable or difficult to find, and this may lead researchers to discard projects in promising areas. Covariance restrictions could enable research to progress in such settings. Finally, covariance restrictions could be employed in conjunction with instruments in many applications. One example is the case of the

[17] Ciliberto et al. (2016) partially identify a correlation coefficient of $Cor(\xi, \eta) \in [0.38, 0.40]$ based on similar data from 2012, and this potentially calls into question the reasonableness of the uncorrelatedness assumption in the airlines industry. Alternatively, their result could be an artifact of the demand specification, which does not incorporate fixed effects.

random coefficients logit demand, for which the covariance restriction could be used to estimate the price parameter *conditional* on the nonlinear demographic parameters, while instruments that vary consumer choice sets could serve to identify the nonlinear demographic parameters. Another example would arise if the covariance restriction allows for over-identification, in which specification tests become available to the econometrician.

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A Linear Models of Supply and Demand

In this appendix we recast the monopoly model of Section 2 in terms of supply and demand, and provide and alternative proof for Proposition 2 that builds explicitly on Hayashi's (2000, chapter 3) canonical textbook treatment of simultaneous equation bias in supply and demand models. We then develop the case of perfect competition with linear demand and marginal costs—which has many similarities to monopoly and one critical difference. The model was a primary focus of previous articles addressing demand identification using covariance restrictions (e.g., Koopmans et al. (1950); Hausman and Taylor (1983); Matzkin (2004)).

A.1 Intuition from Simultaneous Equations: A Link to Hayashi

To start, given the first order conditions of the monopolist, $p_t + (\frac{dq}{dp})^{-1} q_t = \gamma + \eta_t$ for $\frac{dq}{dp} = \beta$, equilibrium in the model can be characterized as follows:

$$\begin{aligned} q_t^d &= \alpha + \beta p_t + \xi_t && \text{(demand)} \\ q_t^s &= \beta\gamma - \beta p_t + \nu_t && \text{(supply)} \\ q_t^d &= q_t^s && \text{(equilibrium)} \end{aligned} \tag{A.1}$$

where $\nu_t \equiv \beta\eta_t$. The only distinction between this model and that of Hayashi is that slope of the supply schedule is determined (solely) by the price parameter of the demand equation, rather than by the increasing marginal cost schedules of perfect competitors.[18]

If market power is the reason that the supply schedule slopes upwards, as it is with our monopoly example, then uncorrelatedness suffices for identification because the model fully pins down how firms adjust prices with demand shocks. Repeating the steps of Hayashi, we have:

$$\beta^{OLS} \equiv plim\left(\hat{\beta}^{OLS}\right) = \beta\left(\frac{Var(\nu) - Var(\xi)}{Var(\nu) + Var(\xi)}\right) \tag{A.2}$$

If variation in the data arises solely due to cost shocks (i.e., $Var(\xi) = 0$) then the OLS estimator is consistent for β. If instead variation arises solely due to demand shocks (i.e., $Var(\nu) = 0$) then the OLS estimator is consistent for $-\beta$. A third special case arises if the demand and cost shocks have equal variance (i.e., $Var(\nu) = Var(\xi)$). Then $\beta^{OLS} = 0$, exactly halfway between the demand slope (β) and the supply slope ($-\beta$). Thus the adjustment required to bring the OLS coefficient in line with either the demand or supply slope is maximized, in terms of absolute value.

It is when variation in the data arises due both cost and demand shocks that the OLS estimate is difficult to interpret. With uncorrelatedness, however, the OLS residuals provide the information required to correct bias. A few lines of algebra obtain:

Lemma A.1. *Under uncorrelatedness, we have*

$$\beta^2 = \left(\beta^{OLS}\right)^2 + \frac{Cov(q, \xi^{OLS})}{Var(p)}. \tag{A.3}$$

and

$$Cov(q, \xi^{OLS}) = \frac{Var(\nu)Var(\xi)}{Var(\nu) + Var(\xi)}. \tag{A.4}$$

[18] A implication of equation (A.1) is that it can be possible to estimate demand parameters by estimating the *supply-side* of the model, taking as given the demand system and the nature of competition. We are aware of precisely one article that employs such a method: Thomadsen (2005) estimates a model of price competition among spatially-differentiated duopolists with (importantly) constant marginal costs.

Proof: See appendix D.

The first equation is a restatement of Proposition 2. The second equation expresses the correction term as function of $Var(\nu)$ and $Var(\xi)$. Notice that the correction term equals zero if variation in the data arises solely due to either cost or demand shocks—precisely the cases for which OLS estimator obtains β and $-\beta$, respectively. Further, the correction term is maximized if $Var(\nu) = Var(\xi)$ which, as developed above, is when the largest adjustment is required because $\beta^{OLS} = 0$.

A.2 Perfect Competition

As a point of comparison, consider perfect competition with linear demand and supply curves. The model is used elsewhere to illustrate the identifying power of covariance restrictions (e.g., Koopmans et al. (1950); Hausman and Taylor (1983); Matzkin (2004)). Let marginal costs be given by $mc = x'\gamma + \lambda q + \eta$. Firms are price-takers and each has a first order condition given by $p = x'\gamma + \lambda q + \eta$. The firm-specific supply curve is $q^s = -\frac{1}{\lambda}x'\gamma + \frac{1}{\lambda}p - \frac{\eta}{\lambda}$. Aggregating across firms and assuming with linearity in demand, we have the following market-level system of equations:

$$\begin{aligned} Q^D &= \beta p + x'\alpha + \xi && \text{(Demand)} \\ Q^S &= \frac{J}{\lambda}p - \frac{J}{\lambda}x'\gamma - \frac{J}{\lambda}\eta && \text{(Supply)} \\ Q^D &= Q^S && \text{(Equilibrium)} \end{aligned} \tag{A.5}$$

where Q^D and Q^S represent market quantity demanded and supplied, respectively. The supply slope depends on the number of firms and the slope of the marginal costs—in stark relief to the monopoly problem in which the supply slope was fully determined by the demand parameter (equation (A.1)).

In this setting, uncorrelatedness allows for the consistent estimation of the price coefficient, but only if the supply slope $\frac{J}{\lambda}$ is known. This mimics our result for oligopoly with constant marginal costs, which, in the limit of perfect competition, yields a flat supply curve. Hausman and Taylor (1983) propose the following methodology: (i) estimate the supply-schedule using an exclusion restriction $\gamma_{[k]} = 0$ for some k; (ii) recover estimates of the supply-side shock; (iii) use these estimated supply-side errors as instruments in demand estimation. Under uncorrelatedness these supply-side errors are orthogonal to demand-shock. (Though it is now understood that a method-of-moments estimator that combines uncorrelatedness with the exclusion restriction would be more efficient.) Matzkin (2004) proposes a similar procedure but relaxes the assumption of linearity.

It is possible demonstrate identification using the methods developed above for models with market power. Indeed, this can be seen as an extension of Corollary 3 because Cournot converges to perfect competition as $J \to \infty$. The OLS estimation of demand yields:

$$\beta^{OLS} \equiv plim(\hat{\beta}^{OLS}) = \beta + \frac{Cov(\xi, p^*)}{Var(p^*)}$$

Tracing the steps provided in Section 2 for the monopoly model, uncorrelatedness implies

$$Cov(\xi, Q) = Cov(\xi^{OLS}, Q) + \frac{\lambda}{J}\frac{Cov(\xi, Q)}{Var(p^*)}Cov(p^*, Q)$$

where ξ^{OLS} is a vector of OLS residuals. Solving for $Cov(\xi, Q)$ and plugging into the probability limit of the OLS estimator yields

$$\beta = \beta^{OLS} - \frac{1}{\frac{J}{\lambda} - \beta^{OLS}}Cov(\xi^{OLS}, Q) \tag{A.6}$$

It follows that β is point identified if the supply slope $\frac{J}{\lambda}$ is known. With an exclusion restriction, $\gamma_{[k]} = 0$, an estimator could be developed using equation (A.6). It would be asymptotically equivalent to the Hausman and Taylor (1983) estimator, and less efficient than the corresponding method-of-moments estimator.

B Generality of Demand

The demand system of equation (5) is sufficiently flexible to nest monopolistic competition with linear demands (e.g., as in the motivating example) and the discrete choice demand models that support much of the empirical research in industrial organization. We illustrate with some typical examples:

1. *Nested logit demand:* Following the exposition of Cardell (1997), let the firms be grouped into $g = 0, 1, \ldots, G$ mutually exclusive and exhaustive sets, and denote the set of firms in group g as $\mathscr{J}_g$. An outside good, indexed by $j = 0$, is the only member of group 0. Then the left-hand-side of equation (5) takes the form

$$h(q_{jt}; w_{jt}) \equiv \ln(q_{jt}/q_{0j}) - \sigma \ln(\bar{s}_{j|g,t})$$

   where $\bar{s}_{j|g,t} = \sum_{j \in \mathscr{J}_g} \frac{q_{jt}}{\sum_{j \in \mathscr{J}_g} q_{jt}}$ is the market share of firm j within its group. The parameter, $\sigma \in [0, 1)$, determines the extent to which consumers substitution disproportionately among firms within the same group. Our second application, developed in Section 5.2, examines this model. If the uncorrelatedness is combined with a supplemental moment, then the full set of parameters can be recovered.

   - add h' for nested logit.

2. *Random coefficients logit demand:* Modifying slightly the notation of Berry (1994), let the indirect utility that consumer $i = 1, \ldots, I$ receives from product j be

$$u_{ij} = \beta p_j + x_j' \alpha + \xi_j + \left[\sum_k x_{jk} \sigma_k \zeta_{ik} \right] + \epsilon_{ij}$$

   where x_{jk} is the kth element of x_j, ζ_{ik} is a mean-zero consumer-specific demographic characteristic, and ϵ_{ij} is a logit error. We have suppressed market subscripts for notational simplicity. Decomposing the RHS of the indirect utility equation into $\delta_j = \beta p_j + x_j' \alpha + \xi_j$ and $\mu_{ij} = \sum_k x_{jk} \sigma_k \zeta_{ik}$, the probability that consumer i selects product j is given by the standard logit formula

$$s_{ij} = \frac{\exp(\delta_j + \mu_{ij})}{\sum_k \exp(\delta_k + \mu_{ik})}$$

   Integrating yields the market shares: $s_j = \frac{1}{I} \sum_i s_{ij}$. Berry et al. (1995) prove that a contraction mapping recovers, for any candidate parameter vector $\tilde{\sigma}$, the vector $\delta(s, \tilde{\sigma})$ that equates these market shares to those observed in the data. This "mean valuation" is $h(s_j; \tilde{\sigma})$ in our notation. The three-step estimator can be applied to recover the price coefficient, again taking as given $\tilde{\sigma}$. This requires an expression for $h'(s, \tilde{\sigma})$, which takes the form

$$h'(s_j; \tilde{\sigma}) = \frac{1}{\frac{1}{I} \sum_i s_{ij}(1 - s_{ij})}.$$

   Thus, uncorrelatedness assumption can recover the linear parameters given the candidate parameter vector $\tilde{\sigma}$. The identification of σ is a distinct issue that has received a great deal of attention from theoretical and applied research (e.g., Romeo (2010); Berry and Haile (2014); Gandhi and Houde (2015); Miller and Weinberg (2017)).

3. *Constant elasticity demand:* With a substitution of $f(p_{jt})$ for p_{jt} into equation (5), the constant elasticity of substitution (CES) demand model of Dixit and Stiglitz (1977) also can be incorporated:

$$\ln(q_{jt}/q_t) = \alpha + \beta \ln\left(\frac{p_{jt}}{\Pi_t}\right) + \xi_{jt}$$

where q_t is an observed demand shifter, Π_t is a price index, and β provides the constant elasticity of demand. This model is often used in empirical research on international trade and firm productivity (e.g., De Loecker (2011); Doraszelski and Jaumandreu (2013)). Due to the constant elasticity, profit-maximization generates $Cov(p, \xi) = 0$, and OLS produces unbiased estimates of the demand parameters. Indeed, this is an excellent illustration of our basic argument: so long as the data generating process is sufficiently well understood, it is possible to characterize the bias of OLS estimates. We opt to focus on semi-linear demand throughout this paper for analytical tractability.

Some demand systems are more difficult to reconcile with equation (5). Consider the linear demand system, $q_{jt} = \alpha_j + \sum_k \beta_{jk} p_k + \xi_{jt}$, which sometimes appears in identification proofs (e.g., Nevo (1998)) but is seldom applied empirically due to the large number of price coefficients. In principle, the system could be formulated such that $h(q_{jt}; w_{jt}) \equiv q_{jt} - \sum_{k \neq j} \beta_{jk} p_k$ and uncorrelatedness assumptions could be used to identify the β_{jj} and α_j coefficients. This would require, however, that the econometrician have other sources of identification for the β_{jk} ($j \neq k$) coefficients, which seems unlikely. The same problem arises with the almost ideal demand system of Deaton and Muellbauer (1980).

C Two-Step Estimation

In the presence of an additional restriction, we can produce a more precise estimator that can be calculated in one fewer step. When the observed cost and demand shifters are uncorrelated, there is no need to project the price on demand covariates when constructing a consistent estimate, and one can proceed immediately using the OLS regression. We formalize the additional restriction and the estimator below.

Assumption 5: *Let the parameters $\alpha^{(k)}$ and $\gamma^{(k)}$ correspond to the demand and supply coefficients for covariate k in X. For any two covariates k and l, $Cov(\alpha^{(k)}x^{(k)}, \gamma^{(l)}x^{(l)}) = 0$.*

Proposition C.1. *Under assumptions 1-3 and 5, a consistent estimate of the price parameter β is given by*

$$\hat{\beta}^{\text{2-Step}} = \frac{1}{2}\left(\hat{\beta}^{OLS} - \frac{\hat{Cov}\left(p, h'(q)q\right)}{\hat{Var}(p)} - \sqrt{\left(\hat{\beta}^{OLS} + \frac{\hat{Cov}\left(p, h'(q)q\right)}{\hat{Var}(p)}\right)^2 + 4\frac{\hat{Cov}\left(\hat{\xi}^{OLS}, h'(q)q\right)}{\hat{Var}(p)}}\right) \tag{C.1}$$

when the auxiliary condition, $\beta < \frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)} - \frac{Cov\left(p^*, h'(q)q\right)}{Var(p^*)}$, holds.

The estimator can be expressed entirely in terms of the data, the OLS coefficient, and the OLS residuals. The first step is an OLS regression of $h(q;\cdot)$ on p and x, and the second step is the construction of the estimator as in equation (C.1). Thus, we eliminate the step of projecting p on x. This estimator will be consistent under the assumption that any covariate affecting demand does not covary with marginal cost. The auxiliary condition parallels that of the three-step estimator, and we expect that it will hold in typical cases.

D Proofs

Proof of Proposition 3 (Set Identification)

From the text, we have $\hat{\beta}^{OLS} \xrightarrow{p} \beta + \frac{Cov(p^*,\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = c + \mu$, where c is the marginal cost and μ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c = X\gamma + \eta$. If we substitute the first-order condition $p^* = X\gamma + \eta + \mu - \hat{p}$ into the bias term from the OLS regression, we obtain

$$\begin{aligned}\frac{Cov(p^*,\xi)}{Var(p^*)} &= \frac{Cov(\xi, X\gamma + \eta + \mu - \hat{p})}{Var(p^*)} \\ &= \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov(\xi,\mu)}{Var(p^*)}\end{aligned}$$

where the second line follows from the exogeneity assumption ($E[X\xi] = 0$). Under our demand assumption, the unobserved demand shock may be written as $\xi = h(q) - x\alpha - \beta p$. At the probability limit of the OLS estimator, we can construct the unobserved demand shock as $\xi = \xi^{OLS} + \left(\beta^{OLS} - \beta\right) p^*$.[19] From the prior step in this proof, $\beta^{OLS} - \beta = \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov(\xi,\mu)}{Var(p^*)}$. Therefore, $\xi = \xi^{OLS} + \left(\frac{Cov(\eta,\xi)}{Var(p^*)} + \frac{Cov(\mu,\xi)}{Var(p^*)}\right) p^*$. This implies

$$\begin{aligned}\frac{Cov\left(\xi,\mu\right)}{Var(p^*)} &= \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \left(\frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\right)\frac{Cov(p^*,\mu)}{Var(p^*)} \\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)}\left(1 - \frac{Cov(p^*,\mu)}{Var(p^*)}\right) &= \frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)} \\ \frac{Cov\left(\xi,\mu\right)}{Var(p^*)} &= \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\mu)}{Var(p^*)}\end{aligned}$$

When we substitute this expression in for β^{OLS}, we obtain

$$\begin{aligned}\beta^{OLS} &= \beta + \frac{Cov(\xi,\eta)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)} \\ \beta^{OLS} &= \beta + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},\mu\right)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}\end{aligned}$$

Thus, we obtain an expression for the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks. If the markup can be parameterized in terms of observables and the correlation in unobserved shocks can be calibrated, we have a method to estimate β from the OLS regression. Under our supply and demand assumptions, $\mu = -\frac{1}{\beta}h'(q)q$, and plugging in obtains the first equation of the proposition:

$$\beta^{OLS} = \beta - \frac{1}{\beta + \frac{Cov(p^*,h'(q)q)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)} + \beta\frac{1}{\beta + \frac{Cov(p^*,h'(q)q)}{Var(p^*)}}\frac{Cov(\xi,\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

[19] For a proof, see a subsequent section in the Appendix.

Proof of Proposition 4 (Point Identification)

Part (1). We first prove the sufficient condition, i.e., that under assumptions 1 and 2, β is the lower root of equation (10) if the following condition holds:

$$0 \leq \beta^{OLS} \frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)} \tag{D.1}$$

Consider a generic quadratic, ax^2+bx+c. The roots of the quadratic are $\frac{1}{2a}\left(-b \pm \sqrt{b^2-4ac}\right)$. Thus, if $4ac < 0$ and $a > 0$ then the upper root is positive and the lower root is negative. In equation (10), $a = 1$, and $4ac < 0$ if and only if equation (D.1) holds. Because the upper root is positive, $\beta < 0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\xi, \eta)$. QED.

Part (2). In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

Lemma D.1. *The roots of equation* (10) *are* β *and* $\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}$.

Proof of Lemma D.1. We first provide equation (10) for reference:

$$\begin{aligned} 0 &= \beta^2 \\ &+ \left(\frac{Cov(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right)\beta \\ &+ \left(-\beta^{OLS}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)}\right) \end{aligned}$$

To find the roots, begin by applying the quadratic formula

$$\begin{aligned} (r_1, r_2) &= \frac{1}{2}\left(-B \pm \sqrt{B^2-4AC}\right) \\ &= \frac{1}{2}\Bigg(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\ &\quad \pm\sqrt{\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 4\beta^{OLS}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)}}\Bigg) \\ &= \frac{1}{2}\Bigg[\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\ &\quad \pm\Bigg(\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right) \\ &\quad + 4\beta^{OLS}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} + 4\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)}\Bigg)^{\frac{1}{2}}\Bigg] \\ &= \frac{1}{2}\Bigg(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\ &\quad \pm\sqrt{\left(\beta^{OLS} + \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)} + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right)}\Bigg) \end{aligned} \tag{D.2}$$

Looking inside the radical, consider the first part: $\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)}$

$$\begin{aligned}
&\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)}\\
&=\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi-p^*(\beta^{OLS}-\beta),h'(q)q\right)}{Var(p^*)}\\
&=\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)}-4\frac{Cov(p^*,\xi)}{Var(p^*)}\frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)}\\
&=\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)}-4\left(\frac{Cov(\xi,\eta)}{Var(p^*)}+\frac{Cov(\xi,-\frac{1}{\beta}h'(q)q)}{Var(p^*)}\right)\frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)}\\
&=\left(\beta^{OLS}+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)^2+4\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)}\left(1+\frac{1}{\beta}\frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)}\right)-4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)} \qquad \text{(D.3)}
\end{aligned}$$

To simplify this expression, it is helpful to use the general form for a firm's first-order condition, $p=c+\mu$, where c is the marginal cost and μ is the markup. We can write $p=p^*+\hat{p}$, where $\hat{p}$ is the projection of p onto the exogenous demand variables, X. By assumption, $c=X\gamma+\eta$. It follows that

$$\begin{aligned}
p^* &= X\gamma+\eta+\mu-\hat{p}\\
&= X\gamma+\eta-\frac{1}{\beta}h'(q)q-\hat{p}
\end{aligned}$$

Therefore

$$Cov(p^*,\xi)=Cov(\xi,\eta)-\frac{1}{\beta}Cov(\xi,h'(q)q)$$

and

$$\begin{aligned}
Cov(\xi,h'(q)q) &= -\beta\left(Cov(p^*,\xi)-Cov(\xi,\eta)\right)\\
\frac{Cov(\xi,h'(q)q)}{Var(p^*)} &= -\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)}-\frac{Cov(\xi,\eta)}{Var(p^*)}\right) \qquad \text{(D.4)}
\end{aligned}$$

Returning to equation (D.3), we can substitute using equation (D.4) and simplify:

$$\begin{aligned}
&\left(\beta^{OLS} + \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 4\frac{Cov\,(\xi, h'(q)q)}{Var(p^*)}\left(1 + \frac{1}{\beta}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right) - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\beta^{OLS}\right)^2 + \left(\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 2\beta^{OLS}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&\quad + 4\frac{Cov\,(\xi, h'(q)q)}{Var(p^*)} + 4\frac{1}{\beta}\frac{Cov\,(\xi, h'(q)q)}{Var(p^*)}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - 4\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&\quad - 4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 2\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&\quad - 4\beta\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right) - 4\left(\frac{Cov(p^*,\xi)}{Var(p^*)}\right)\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
&= \beta^2 + \left(\frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right)^2 + 2\beta\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} \\
&\quad - 2\beta\frac{Cov\,(p^*,\xi)}{Var(p^*)} - 2\frac{Cov\,(p^*,\xi)}{Var(p^*)}\frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
&= \left(\left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right) - \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)}
\end{aligned}$$

Now, consider the second part inside of the radical in equation (D.2):

$$\begin{aligned}
&\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right) \\
&= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\beta + \frac{Cov(\xi,\eta)}{Var(p^*)} - \frac{1}{\beta}\frac{Cov(\xi, h'(q)q)}{Var(p^*)} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right) \\
&= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 + 2\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(\xi, h'(q)q)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} \\
&= -\left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right) + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, h'(q)q)}{Var(p^*)}
\end{aligned}$$

Combining yields a simpler expression for the terms inside the radical of equation (D.2):

$$\begin{aligned}
&\left(\left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right) - \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + 4\beta\frac{Cov(\xi,\eta)}{Var(p^*)} \\
&\quad + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\beta - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)}\right) - \frac{Cov\,(p^*,\xi)}{Var(p^*)}\right)^2 + \left(\frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2 \\
&\quad + 2\beta\frac{Cov(\xi,\eta)}{Var(p^*)} - 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*,\xi)}{Var(p^*)} + 2\frac{Cov(\xi,\eta)}{Var(p^*)}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} \\
&= \left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2
\end{aligned}$$

Plugging this back into equation (D.2), we have:

$$\begin{aligned}(r_1, r_2) &= \frac{1}{2}\left(\beta^{OLS} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right.\\ &\left.\pm\sqrt{\left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right)\\ &= \frac{1}{2}\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right.\\ &\left.\pm\sqrt{\left(\beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2}\right)\end{aligned}$$

The roots are given by

$$\frac{1}{2}\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} + \beta + \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov\,(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)$$
$$=\beta$$

and

$$\frac{1}{2}\left(\beta + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta - \frac{Cov\,(p^*, h'(q)q)}{Var(p^*)} + \frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}\right)$$
$$=\frac{Cov\,(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}$$

which completes the proof of the intermediate result. QED.

Part (3). Consider the roots of equation (10), β and $\frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)}$. The price parameter β may or may not be the lower root.[20] However, β is the lower root iff

$$\begin{aligned}
\beta &< \frac{Cov(p^*,\xi)}{Var(p^*)} - \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \beta\frac{Cov(p^*,-\frac{1}{\beta}h'(q)q)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \beta\frac{Cov(p^*,p^*-mc)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
\beta &< \beta\frac{Var(p^*)}{Var(p^*)} - \beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} - \beta\frac{Cov(p^*,\eta)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
0 &< -\beta\frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} - \beta\frac{Cov(p^*,\eta)}{Var(p^*)} - \frac{Cov(\xi,\eta)}{Var(p^*)} \\
0 &< \frac{Cov(p^*,-\frac{1}{\beta}\xi)}{Var(p^*)} + \frac{Cov(p^*,\eta)}{Var(p^*)} + \frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)}
\end{aligned}$$

The third line relies on the expression for the markup, $p - mc = -\frac{1}{\beta}h'(q)q$. The final line holds because $\beta < 0$ so $-\beta > 0$. It follows that β is the lower root of (10) iff

$$-\frac{1}{\beta}\frac{Cov(\xi,\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*,-\frac{1}{\beta}\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\eta\right)}{Var(p^*)}$$

in which case β is point identified given knowledge of $Cov(\xi,\eta)$. QED.

Proof of Proposition 5 (Prior-Free Bounds)

The proof is again an application of the quadratic formula. Any generic quadratic, ax^2+bx+c, with roots $\frac{1}{2}\left(-b\pm\sqrt{b^2-4ac}\right)$, admits no solution in real numbers if $b^2 < 4ac$. Given the formulation of (10), there are no real solutions if

$$\left(\frac{Cov(p^*,h'(q)q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right)^2 < 4\left(-\beta^{OLS}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)}\right)$$

This is equivalent to

$$\pm\left(\frac{Cov(p^*,h'(q)q)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} - \beta^{OLS}\right) < 2\sqrt{\left(-\beta^{OLS}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)}\right)}$$

[20] Consider that the first root is the upper root if

$$\beta + \frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} > 0$$

because, in that case,

$$\sqrt{\left(\beta + \frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2} = \beta + \frac{Cov\left(p^*,h'(q)q\right)}{Var(p^*)} - \frac{Cov\left(p^*,\xi\right)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}$$

When $\beta + \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)} < 0$, then $\sqrt{\left(\beta + \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)^2} = -\left(\beta + \frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(p^*,\xi)}{Var(p^*)} + \frac{Cov(\xi,\eta)}{Var(p^*)}\right)$, and the first root is then the lower root (i.e., minus the negative value).

Solving for $Cov(\xi,\eta)$ we obtain $Cov(\xi,\eta) < c_1$ and $Cov(\xi,\eta) > c_2$, where c_1 and c_2 are defined as

$$\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] = \left[\begin{array}{c} Var(p^*)\beta^{OLS} - Cov(p^*,h'(q)q) - \sqrt{-Var(p^*)\left(Cov(p^*,h'(q)q)\beta^{OLS} + Cov(\xi^{OLS},h'(q)q)\right)} \\ Var(p^*)\beta^{OLS} - Cov(p^*,h'(q)q) + \sqrt{-Var(p^*)\left(Cov(p^*,h'(q)q)\beta^{OLS} + Cov(\xi^{OLS},h'(q)q)\right)} \end{array}\right]$$

These bounds exist in real numbers if the expression inside the radicals is positive, which is the case if and only if the sufficient condition for point identification from Proposition 4 fails. By observation, we have $c_1 < c_2$, so that values of $Cov(\xi,\eta) \in (c_1, c_2)$ can be ruled out. QED.

Proof of Proposition 6 (Non-Constant Marginal Costs)

Under the semi-linear marginal cost schedule of equation (13), the plim of the OLS estimator is equal to

$$\text{plim}\hat{\beta}^{OLS} = \beta + \frac{Cov(\xi,g(q))}{Var(p^*)} - \frac{1}{\beta}\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for p: $Cov(p^*,\xi) = Cov(g(q) + \eta - \frac{1}{\beta}h'(q)q - \hat{p},\xi) = Cov(\xi,g(q)) - \frac{1}{\beta}Cov(\xi,h'(q)q)$under the assumptions. Next, we re-express the terms including the unobserved demand shocks in in terms of OLS residuals. The unobserved demand shock may be written as $\xi = h(q) - x\beta_x - \beta p$. The estimated residuals are given by $\xi^{OLS} = \xi + \left(\beta - \beta^{OLS}\right)p^*$. As $\beta - \beta^{OLS} = \frac{1}{\beta}\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)} - \frac{Cov(\xi,g(q))}{Var(p^*)}$, we obtain $\xi^{OLS} = \xi + \left(\frac{1}{\beta}\frac{Cov\left(\xi,h'(q)q\right)}{Var(p^*)} - \frac{Cov(\xi,g(q))}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\xi^{OLS},h'(q)q\right) = \left(1 + \frac{1}{\beta}\frac{Cov(p^*,h'(q)q)}{Var(p^*)}\right)Cov(\xi,h'(q)q) - \frac{Cov(p^*,h'(q)q)}{Var(p^*)}Cov(\xi,g(q))$$
$$Cov\left(\xi^{OLS},g(q)\right) = \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)}Cov\left(\xi,h'(q)q\right) + \left(1 - \frac{Cov(p^*,g(q))}{Var(p^*)}\right)Cov(\xi,g(q))$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\left[\begin{array}{c} Cov(\xi,h'(q)q) \\ Cov(\xi,g(q)) \end{array}\right] = \left[\begin{array}{cc} 1 + \frac{1}{\beta}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} & -\frac{Cov(p^*,h'(q)q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{array}\right]^{-1} \left[\begin{array}{c} Cov(\xi^{OLS},h'(q)q) \\ Cov(\xi^{OLS},g(q)) \end{array}\right]$$

where

$$\left[\begin{array}{cc} 1 + \frac{1}{\beta}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} & -\frac{Cov(p^*,h'(q)q)}{Var(p^*)} \\ \frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} \end{array}\right]^{-1} =$$
$$\frac{1}{1 + \frac{1}{\beta}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} - \frac{Cov(p^*,g(q))}{Var(p^*)}}\left[\begin{array}{cc} 1 - \frac{Cov(p^*,g(q))}{Var(p^*)} & \frac{Cov(p^*,h'(q)q)}{Var(p^*)} \\ -\frac{1}{\beta}\frac{Cov(p^*,g(q))}{Var(p^*)} & 1 + \frac{1}{\beta}\frac{Cov(p^*,h'(q)q)}{Var(p^*)} \end{array}\right].$$

Therefore, we obtain the relations

$$Cov(\xi, h'(q)q) = \frac{\left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right) Cov(\xi^{OLS}, h'(q)q) + \frac{Cov(p^*, h'(q)q)}{Var(p^*)} Cov(\xi^{OLS}, g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(p^*, g(q))}{Var(p^*)}}$$

$$Cov(\xi, g(q)) = \frac{-\frac{1}{\beta}\frac{Cov(p^*, g(q))}{Var(p^*)} Cov(\xi^{OLS}, h'(q)q) + \left(1 + \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right) Cov(\xi^{OLS}, g(q))}{1 + \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(p^*, g(q))}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\xi, g(q)) - \frac{1}{\beta} Cov(\xi, h'(q)q)$ in the plim of the OLS estimator and simplify:

$$\begin{aligned}
&\left(1 + \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\left(Cov(\xi, g(q)) - \frac{1}{\beta} Cov(\xi, h'(q)q)\right) \\
=& -\frac{1}{\beta}\frac{Cov(p^*, g(q))}{Var(p^*)} Cov(\xi^{OLS}, h'(q)q) + \left(1 + \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)}\right) Cov(\xi^{OLS}, g(q)) \\
& -\frac{1}{\beta}\left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right) Cov(\xi^{OLS}, h'(q)q) - \frac{1}{\beta}\frac{Cov(p^*, h'(q)q)}{Var(p^*)} Cov(\xi^{OLS}, g(q)) \\
=& Cov(\xi^{OLS}, g(q)) - \frac{1}{\beta} Cov(\xi^{OLS}, h'(q)q).
\end{aligned}$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$\text{plim}\hat{\beta}^{OLS} = \beta - \frac{\frac{Cov(\xi^{OLS}, h'(q)q)}{Var(p^*)} - \beta\frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}}{\beta + \frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \beta\frac{Cov(p^*, g(q))}{Var(p^*)}},$$

and the following quadratic β.

$$\begin{aligned}
0 =& \left(1 - \frac{Cov(p^*, g(q))}{Var(p^*)}\right)\beta^2 \\
&+ \left(\frac{Cov(p^*, h'(q)q)}{Var(p^*)} - \hat{\beta}^{OLS} + \frac{Cov(p^*, g(q))}{Var(p^*)}\hat{\beta}^{OLS} + \frac{Cov(\xi^{OLS}, g(q))}{Var(p^*)}\right)\beta \\
&+ \left(-\frac{Cov(p^*, h'(q)q)}{Var(p^*)}\hat{\beta}^{OLS} - \frac{Cov(\xi^{OLS}, h'(q)q)}{Var(p^*)}\right).
\end{aligned}$$

QED.

Proof of Lemma A.1

The proof is by construction. Note that model has the solutions $p_t^* = \frac{1}{2}\left(-\frac{\alpha}{\beta} - \frac{\xi_t}{\beta} + \gamma + \frac{\nu_t}{\beta}\right)$ and $q_t^* = \frac{1}{2}(\alpha + \xi_t + \beta\gamma + \nu_t)$, where again $\nu_t \equiv \beta\eta_t$. The following objects are easily derived:

$$Cov(p, \xi) = -\frac{1}{2\beta} Var(\xi) \qquad Cov(p, \nu) = \frac{1}{2\beta} Var(\nu)2$$

$$Var(p) = \frac{Var(\nu) + Var(\xi)}{(2\beta)^2} \qquad Var(q) = \frac{1}{4}(Var(\xi) + Var(\nu))$$

Using the above, we have

$$\begin{aligned} Cov(p,q) &= Cov(p,\alpha+\beta p+\xi) = \beta Var(p) + Cov(p,\xi) = \beta\frac{Var(\nu)+Var(\xi)}{(2\beta)^2} - \frac{2\beta}{(2\beta)^2}Var(\xi) \\ &= \frac{\beta Var(\nu)+\beta Var(\xi) - 2\beta Var(\xi)}{(2\beta)^2} = \beta\frac{Var(\nu)-Var(\xi)}{(2\beta)^2} \end{aligned}$$

And that obtains equation (A.2):

$$plim\left(\hat{\beta}^{OLS}\right) \equiv \beta^{OLS} = \frac{Cov(p,q)}{Var(p)} = \beta\frac{Var(\nu)-Var(\xi)}{Var(\nu)+Var(\xi)}$$

Equation (A.4) requires an expression for $Cov(q,\xi^{OLS})$. Define

$$plim(\hat{\xi}^{OLS}) \equiv \xi^{OLS} = q - \alpha^{OLS} - \beta^{OLS}p$$

Then, plugging into $Cov(q,\xi^{OLS})$ using the objects derived above, we have

$$\begin{aligned} Cov(q,\xi^{OLS}) &= Cov(q, q-\beta^{OLS}p) \\ &= Var(q) - \beta^{OLS}Cov(p,q) \\ &= \frac{1}{4}(Var(\xi)+Var(\nu)) - \left(\beta\frac{Var(\nu)-Var(\xi)}{Var(\nu)+Var(\xi)}\right)\left(\beta\frac{(Var(\nu)-Var(\xi))}{(2\beta)^2}\right) \\ &= \frac{1}{4}\left(\frac{[Var(\xi)+Var(\nu)]^2 - [Var(\nu)-Var(\xi)]^2}{Var(\nu)+Var(\xi)}\right) \\ &= \frac{Var(\xi)Var(\nu)}{Var(\nu)+Var(\xi)} \end{aligned}$$

We turn now to equation (A.3). Based on the above, we have that

$$\frac{Cov(q,\xi^{OLS})}{Var(p)} = \left(\frac{Var(\xi)Var(\nu)}{Var(\nu)+Var(\xi)}\right)\frac{(2\beta)^2}{Var(\nu)+Var(\xi)} = (2\beta)^2\frac{Var(\xi)Var(\nu)}{[Var(\nu)+Var(\xi)]^2}$$

and now only few more lines of algebra are required:

$$\begin{aligned} (\beta^{OLS})^2 + \frac{Cov(q,\xi^{OLS})}{Var(p)} &= \beta^2\left[\frac{Var(\nu)-Var(\xi)}{Var(\nu)+Var(\xi)}\right]^2 + (2\beta)^2\frac{Var(\xi)Var(\nu)}{[Var(\nu)+Var(\xi)]^2} \\ &= \frac{\beta^2[Var^2(\nu)+Var^2(\xi)-2Var(\nu)Var(\xi)] + 4\beta^2 Var(\nu)Var(\xi)}{[Var(\nu)+Var(\xi)]^2} \\ &= \frac{\beta^2[Var^2(\nu)+Var^2(\xi)+2Var(\nu)Var(\xi)]}{[Var(\nu)+Var(\xi)]^2} \\ &= \beta^2\frac{[Var(\nu)+Var(\xi)]^2}{[Var(\nu)+Var(\xi)]^2} = \beta^2 \end{aligned}$$

QED.

Proof of Proposition 7 (Multi-Product Firms)

Under assumption 4, define $\frac{dw_k}{dp_j} = \beta f_{kj}$. The market subscript, t, is omitted to simplify notation. Let K^m denote the set of products owned by multi-product firm m. When the firm sets prices on each of its products to maximize joint profits, there are $|K^m|$ first-order conditions, which can be expressed as

$$\sum_{k\in K^m}(p_k - c_k)\frac{\partial q_k}{\partial p_j} = -q_j \ \ \forall j \in K^m.$$

For demand systems satisfying assumptions 1 and 4,

$$\frac{\partial q_j}{\partial p_j} = \beta\frac{1}{dh/dq_j}\left(1-\frac{dh}{dw_j}f_{jj}\right)$$

and

$$\frac{\partial q_k}{\partial p_j} = -\beta\frac{1}{dh/dq_k}\frac{dh}{dw_k}f_{kj}$$

Therefore, the set of first-order conditions can be written as

$$\sum_{k\in K^m}(p_k - c_k)\frac{1}{\partial h/\partial q_k}\left(\mathbf{1}[j=k]-\frac{\partial h}{\partial w_k}f_{kj}\right) = -\frac{1}{\beta}q_j \ \ \forall j \in K^m.$$

Stack the first-order conditions, writing the LHS as the product of a vector of markups $(p_j - c_j)$ and a matrix A^m of loading components, $A^m_{i(j),i(k)} = \frac{1}{dh/dq_k}(\mathbf{1}[j=k] - \frac{dh}{dw_k}f_{kj})$, where $i(\cdot)$ indexes products within a firm. Next, invert the loading matrix to solve for markups as function of the loading components and $-\frac{1}{\beta}\boldsymbol{q}^m$, where $\boldsymbol{q}^m$ is a vector of the multi-product firm's quantities. Equilibrium prices equal costs plus a markup, where the markup is determined by the inverse of A^m ($(A^m)^{-1} \equiv \Lambda^m$), quantities, and the price parameter:

$$p_j = c_j - \frac{1}{\beta}\left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}. \quad \text{(D.5)}$$

Here, $\left(\Lambda^m \boldsymbol{q}^m\right)_{i(j)}$ provides the entry corresponding to product j in the vector $\Lambda^m\boldsymbol{q}^m$. As the matrix Λ^m is not a function of the price parameter after conditioning on observables, this form of the first-order condition allows us to solve for β using a quadratic three-step solution analogous to that in equation (2).[21] With the modified first-order conditions of equation (D.5), the quadratic in the proposition can be derived following the proof of Proposition 2. QED.

Proof of Proposition C.1 (Two-Step Estimator)

Suppose that, in addition to assumptions 1-3, that marginal costs are uncorrelated with the exogenous demand factors (Assumption 5). Then, the expression $\frac{1}{\beta+\frac{Cov(p,h'(q)q)}{Var(p)}}\frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p)}$ is equal to $\frac{1}{\beta+\frac{Cov(p^*,h'(q)q)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS},h'(q)q\right)}{Var(p^*)}$.

[21] At this point, the reader may be wondering where the prices of other firms are captured under the adjusted first-order conditions for multi-product ownership. As is the case with single product firms, we expect prices of other firm's products to be included in w_j, which is appropriate under Bertrand price competition.

Assumption 4 implies $Cov(\hat{p}, c) = 0$, allowing us to obtain

$$\begin{aligned}
Cov(\hat{p}, \beta(\hat{p} + p^* - c)) &= \beta Var(\hat{p}) \\
Cov(p - p^*, \beta(\hat{p} + p^* - c)) &= \beta Var(p) - \beta Var(p^*) \\
Var(p)\beta + Cov\left(p, h'(q)q\right) &= Var(p^*)\beta + Cov\left(p^*, h'(q)q\right) \\
\left(\beta + \frac{Cov\left(p, h'(q)q\right)}{Var(p)}\right)\frac{1}{Var(p^*)} &= \left(\beta + \frac{Cov\left(p^*, h'(q)q\right)}{Var(p^*)}\right)\frac{1}{Var(p)} \\
\frac{1}{\beta + \frac{Cov(p^*, h'(q)q)}{Var(p^*)}}\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p^*)} &= \frac{1}{\beta + \frac{Cov(p, h'(q)q)}{Var(p)}}\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p)}.
\end{aligned}$$

Therefore, the probability limit of the OLS estimator can be written as:

$$\text{plim}\hat{\beta}^{OLS} = \beta - \frac{1}{\beta + \frac{Cov(p, h'(q)q)}{Var(p)}}\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p)}.$$

The roots of the implied quadratic are:

$$\frac{1}{2}\left(\beta^{OLS} - \frac{Cov\left(p, h'(q)q\right)}{Var(p)} \pm \sqrt{\left(\beta^{OLS} + \frac{Cov\left(p, h'(q)q\right)}{Var(p)}\right)^2 + 4\frac{Cov\left(\xi^{OLS}, h'(q)q\right)}{Var(p)}}\right)$$

which are equivalent to the pair $\left(\beta, \beta\left(1 - \frac{Var(p^*)}{Var(p)}\right) + \frac{Cov(p^*, \xi)}{Var(p^*)} - \frac{Cov\left(p^*, h'(q)q\right)}{Var(p)}\right)$. Therefore, with the auxiliary condition $\beta < \frac{Cov(p^*, \xi)}{Var(p^*)}\frac{Var(p)}{Var(p^*)} - \frac{Cov\left(p^*, h'(q)q\right)}{Var(p^*)}$, the lower root is consistent for β. QED.

E A Consistent and Unbiased Estimate for ξ

The following proof shows a consistent and unbiased estimate for the unobserved term in a linear regression when one of the covariates is endogenous. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace p with $\ln p$ everywhere and obtain the same results.

We can construct both the true demand shock and the OLS residuals as:

$$\begin{aligned}
\xi &= h(q) - \beta p - x'\alpha \\
\xi^{OLS} &= h(q) - \beta^{OLS} p - x'\alpha^{OLS}
\end{aligned}$$

where this holds even in small samples. Without loss of generality, we assume $E[\xi] = 0$. The true demand shock is given by $\xi_0 = \xi^{OLS} + (\beta^{OLS} - \beta)p + x'(\alpha^{OLS} - \alpha)$. We desire to show that an alternative estimate of the demand shock, $\xi_1 = \xi^{OLS} + (\beta^{OLS} - \beta)p^*$, is consistent and unbiased. (This eliminates the need to estimate the true α parameters). It suffices to show that $(\beta^{OLS} - \beta)p^* \to (\beta^{OLS} - \beta)p + x'(\alpha^{OLS} - \alpha)$. Consider the projection matrices

$$\begin{aligned}
Q &= I - P(P'P)^{-1}P' \\
M &= I - X(X'X)^{-1}X',
\end{aligned}$$

where P is an $N \times 1$ matrix of prices and X is the $N \times k$ matrix of covariates x. Denote $Y \equiv h(q) =$

$P\beta + X\alpha + \xi$. Our OLS estimators can be constructed by a residualized regression

$$\alpha^{OLS} = ((XQ)'QX)^{-1} (XQ)' Y$$
$$\beta^{OLS} = ((PM)'MP)^{-1} (PM)' Y.$$

Therefore

$$\begin{aligned}\alpha^{OLS} &= (X'QX)^{-1} (X'QP\beta + X'QX\alpha + X'Q\xi) \\ &= \alpha + (X'QX)^{-1} X'Q\xi.\end{aligned}$$

Similarly,

$$\begin{aligned}\beta^{OLS} &= (P'MP)^{-1} (P'MP\beta + P'MX\alpha + P'M\xi) \\ &= \beta + (P'MP)^{-1} P'M\xi.\end{aligned}$$

We desire to show

$$P^*(\beta^{OLS} - \beta) \to P(\beta^{OLS} - \beta) + X(\alpha^{OLS} - \alpha).$$

Note that $P^* = MP$. Then

$$\begin{aligned}P^*(\beta^{OLS} - \beta) &\to P(\beta^{OLS} - \beta) + X(\alpha^{OLS} - \alpha) \\ MP (P'MP)^{-1} P'M\xi &\to P (P'MP)^{-1} P'M\xi + X (X'QX)^{-1} X'Q\xi \\ -X(X'X)^{-1}X'P (P'MP)^{-1} P'M\xi &\to X (X'QX)^{-1} X'Q\xi \\ -X(X'X)^{-1}X'P (P'MP)^{-1} P' \left[I - X(X'X)^{-1}X'\right] \xi &\to X (X'QX)^{-1} X' \left[I - P(P'P)^{-1}P'\right] \xi \\ -X(X'X)^{-1}X'P (P'MP)^{-1} P'\xi &\to X (X'QX)^{-1} X'\xi \\ +X(X'X)^{-1}X'P (P'MP)^{-1} P'X(X'X)^{-1}X'\xi & \quad - X (X'QX)^{-1} X'P(P'P)^{-1}P'\xi.\end{aligned}$$

We will show that the following two relations hold, which proves consistency and completes the proof.

1. $X(X'X)^{-1}X'P (P'MP)^{-1} P'\xi = X (X'QX)^{-1} X'P(P'P)^{-1}P'\xi$
2. $X(X'X)^{-1}X'P (P'MP)^{-1} P'X(XX (X'QX)^{-1} X'X)^{-1}X'\xi \to X (X'QX)^{-1} X'\xi$

Part 1: Equivalence

It suffices to show that $X(X'X)^{-1}X'P\,(P'MP)^{-1} = X\,(X'QX)^{-1}\,X'P(P'P)^{-1}$.

$$
\begin{aligned}
X(X'X)^{-1}X'P\,(P'MP)^{-1} =& X\,(X'QX)^{-1}\,X'P(P'P)^{-1} \\
X(X'X)^{-1}X'P =& X\,(X'QX)^{-1}\,X'P(P'P)^{-1}\,(P'MP) \\
X(X'X)^{-1}X'P =& X\,(X'QX)^{-1}\,X'P(P'P)^{-1}\,(P'P) \\
&- X\,(X'QX)^{-1}\,X'P(P'P)^{-1}\left(P'X(X'X)^{-1}X'P\right) \\
X(X'X)^{-1}X'P =& X\,(X'QX)^{-1}\,X'P \\
&- X\,(X'QX)^{-1}\,X'\,[I-Q]\,X(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X\,(X'QX)^{-1}\,X'P \\
&- X\,(X'QX)^{-1}\,X'X(X'X)^{-1}X'P \\
&+ X\,(X'QX)^{-1}\,X'QX(X'X)^{-1}X'P \\
X(X'X)^{-1}X'P =& X(X'X)^{-1}X'P
\end{aligned}
$$

QED.

Part 2: Consistency (and Unbiasedness)

Because $X(X'X)^{-1}X'P = X\,(X'QX)^{-1}\,X'P(P'P)^{-1}\,(P'MP)$, as shown above:

$$
\begin{aligned}
X(X'X)^{-1}X'P\,(P'MP)^{-1}\,P'X(X'X)^{-1}X'\xi &\to X\,(X'QX)^{-1}\,X'\xi \\
X\,(X'QX)^{-1}\,X'P(P'P)^{-1}P'X(X'X)^{-1}X'\xi &\to X\,(X'QX)^{-1}\,X'\xi \\
X\,(X'QX)^{-1}\,X'\,[I-Q]\,X(X'X)^{-1}X'\xi &\to X\,(X'QX)^{-1}\,X'\xi \\
X\,(X'QX)^{-1}\,X'X(X'X)^{-1}X'\xi &\to X\,(X'QX)^{-1}\,X'\xi \\
-X(X'X)^{-1}X'\xi & \\
X\,(X'QX)^{-1}\,X'\xi - X(X'X)^{-1}X'\xi &\to X\,(X'QX)^{-1}\,X'\xi \\
X(X'X)^{-1}X'\xi &\to 0.
\end{aligned}
$$

The last line, where the projection of ξ onto the exogenous covariates X converges to zero, holds by assumption. We say that two vectors converge if the mean absolute deviation goes to zero as the sample size gets large. Note that also $E[X(X'X)^{-1}X'\xi] = 0$, so ξ_1 is both a consistent and unbiased estimate for ξ_0. QED.