# Market Design and Walrasian Equilibrium†

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**Abstract**

We establish the existence of Walrasian equilibrium for economies with many discrete goods and possibly one divisible good. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or class assignment. We consider all economies with quasilinear gross substitutes preferences. We allow agents to have limited quantities of the divisible good (constrained economies). We also consider economies without a divisible good (no transfer payments economies). In the latter case, we assume that the seller (i.e., market designer) initially owns all of the goods. For the constrained case, we show the existence and efficiency of Walrasian equilibrium. For the no transfer payments case, we show the existence and efficiency of strong (Walrasian) equilibrium. We also show that aggregate constraints of the kind that are relevant for school choice/class assignment problems can be accommodated by either incorporating these constraints into individuals preference or by incorporating a suitable production technology into no transfer payments economies.

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## 1. Introduction

In this paper, we establish the existence of Walrasian equilibrium in economies with many discrete goods and either with a limited quantity of one divisible good or without any divisible goods. Our goal is not only to study Walrasian equilibria in new settings but also to facilitate the use of market mechanisms in resource allocation problems such as school choice or class assignment. To this end, we develop techniques for analyzing allocation problems in economies with or without transfers and for incorporating additional constraints into allocation rules.

In Kelso and Crawford (1982)'s formulation of this problem, there is a finite number of goods, and a finite number of consumers with quasilinear utility functions that satisfy the *substitutes property.* Kelso and Crawford also assume that each consumer is endowed with enough of the divisible good to ensure that she can purchase any bundle of discrete goods at the equilibrium prices. This last condition would be satisfied, for example, if each consumer had more money than the value she assigns to the aggregate endowment of indivisible commodities. We call the Kelso-Crawford setting the *unconstrained economy.* Kelso and Crawford's ingenious formulation of the substitutes condition facilitates their existence theorem as well as a tatonnement process/dynamic auction for computing Walrasian equilibrium. Subsequent research has identified various important properties of Walrasian equilibrium in unconstrained economies.

Our goal is to do away with the assumption that each consumer has enough of the divisible goods to purchase whatever she may wish at the equilibrium prices. In particular, we will allow for arbitrary positive endowments of the divisible good. We call this the *constrained economy.* We also consider the *no transfers payments* economy; that is, we consider economies in which all goods are initially owned by the market designer and there is no divisible good. This setting is particularly well-suited for the analysis of many allocation problems such as school choice or class selection.

Substitutes preferences have been used to analyze a variety of market design problems. Results suggesting that the Walras equilibrium correspondence is nearly incentive compatible when there are sufficiently many agents (Roberts and Postlewaite, (1976)) have been invoked to argue that Walrasian methods can play a role in market design. In most of these applications it is unreasonable to assume that each agent has enough of the divisible good to acquire whatever she wishes. In many applications, transfers (i.e., money) are ruled out altogether and the problem is one of assigning efficiently and fairly a fixed number of objects to individuals. Hence, both the constrained economy and the no transfer payments economy are of interest.

Theorem 1 establishes the existence of a Walrasian equilibrium (henceforth, equilibrium) in random allocations for constrained economies. In the unconstrained case, randomization is not necessary since in such economies, a random equilibrium allocation at prices $p$ is simply a probability distribution over deterministic competitive equilibria at prices $p$. However, in both constrained and no transfer payments economies, randomization is necessary for the existence of equilibrium. As an illustration, consider the following example.

**Example 1:** Students are required to take at least 3 interdisciplinary classes, each combining two fields of study. No two classes can combine the same two fields and all four fields must be part of some class. We also assume free disposal to guarantee monotonicity, that is, students can freely take a subset of classes they have access to. Let $a, b, c, d$ be the four fields of study. Interdisciplinary course comes in two varieties, the generic varieties $C = \{ab, ac, ad, bc, bd, cd\}$ and three classes taught by a star teacher $C^* = \{ab^*, ac^*, ad^*\}$. The study plan $A = \{ab^*, ac, cd\}$ would meet the constraint but the plan $B = \{ab, bc, ac\}$ would not since it omits field $d$. Similarly, the plan $B' = \{ab, ab^*, cd\}$ would violate the constraint since it involves the same type of class twice. For simplicity, we assume that each class serves exactly one student. The economy has three agents, agents 1 and 2 are students while agent 3 is not a student. If a student chooses a course schedule $A$ that satisfies the constraint described above then the student's utility from that schedule is $u(A) = 2|A \cap C^*|$. If a student violates the constraint, her utility is $u(A) = -\infty$. We can show that $u$ is monotone and satisfies the substitutes property defined in Section 2.1[1]. Initially, student 1 is endowed with two units of money and the schedule $\{ab, bc, cd\}$; student 2 is similarly endowed with two units of money and the schedule $\{ac, ad, bd\}$. Agent

[1] The proof can be found in Appendix A.

3 derives no utility from classes and cares only about money. Agent 3's initial endowment is $\{ab^*, ac^*, ad^*\}$. Utility functions are quasilinear in money and have the form

$$U_i(A, p) = u_i(A) - p(A)$$

where $p(A) = \sum_{k \in A} p^k$ is the price of schedule $A$.

This economy has no deterministic equilibrium. To see this, note that efficiency requires that the two students consume the three starred classes. Thus, in any efficient deterministic allocation one of the two students must attend two classes from $C^*$ while the other attends one class in that set. Since generic classes are in abundant supply, their prices in any candidate equilibrium must be zero. On the other hand, prices for non-generic classes must be strictly positive and $p^j + p^k \leq 2$ for some pair in the set $C^*$. But under those conditions, the equilibrium utility of the student who chooses 2 non-generic classes would exceed the equilibrium utility of the student who consumes one, which cannot be the case in competitive equilibrium since the two students have the same utility function and budget set.

While there are no deterministic equilibria, there exists an equilibrium with *random allocations.* Specifically, $p^k = 4/3, k \in C^*$ and $p^k = 0, k \in C$, together with the random allocation in which each student chooses the schedules $A = \{ab^*, ac^*, cd\}$ and $B = \{bc, bd, ad^*\}$ with equal probabilities is an equilibrium. Notice that, in equilibrium, each student demands $1/2$ probability of each good in the set $A \cup B$. However, this marginal distribution does not convey enough information to determine the student's consumption. Thus, we must describe random consumption as a lottery over *consumption bundles.*[2]

In a randomized equilibrium, feasibility requires that (1) the expected aggregate consumption of each good is smaller than or equal to one and that (2) there exists a lottery that implements the desired random demands. In the example above, we found an equilibrium that satisfies both conditions. However, in the same example there are optimal demands that satisfy (1) but not (2). For example, suppose student 1's demand is $\{ab^*, ac^*, cd\}$ and $\{ad^*, bc, bd\}$ with equal probabilities while student 2's demand is $\{ab^*, ad^*, cd\}$ and

[2] Budish et. al. (2013) consider a given marginal distribution over consumptions but allow for constraints on the realized demand. However, their hierarchical constraints cannot capture the constraints necessary to capture optimal plans in this example.

$\{ac^*, bc, bd\}$ with equal probabilities. Clearly, these demands satisfy (1) but there is no lottery that can implement them. Thus, the implementability of a random allocation is an essential part of the existence problem.

Example 1, above, features utilities that satisfy the substitutes property and, as a result, there exists a competitive equilibrium. Example 2, below, illustrates how equilibrium existence fails if utilities do not satisfy the substitutes property.

**Example 2:** The economy has three agents and three indivisible goods (henceforth goods). Initially, agents 1 and 2 each have 1 unit of money (the divisible good) and no goods. Agent 3's initial endowment consists of the three goods and zero units of money. For agents 1 and 2,

$$u_i(A) = \begin{cases} 0 & \text{if } |A| < 2 \\ 2 & |A| \geq 2 \end{cases}$$

while $u_3(A) = 0$ for all $A$. Since the three goods are perfect substitutes, in equilibrium, all three must have the same price. Let $r$ be this common price. Clearly, $r = 0$ is impossible in any equilibrium since both agents 1 and 2 would demand at least 2 goods with probability 1 and market clearing fails. Then in any equilibrium, $r > 0$ and agents 1 and 2 must consume the three goods with probability 1. This implies $r \cdot 3 \leq 2$ and hence $r \leq 2/3$. At $r \leq 2/3$ both agent 1 and 2 will want to consume any 2 of goods with probability $1/(2r)$ and 0 goods with probability $1 - /(2r)$. Market clearing requires, at a minimum, that the expected total consumption of the these two agents is 3. Hence, $2 \cdot 2 \cdot 1/(2r) = 3$ and therefore $r = 2/3$. This means that the optimal random consumption bundles for agent $i = 1, 2$ is the distribution that assigns her 2 goods with probability $3/4$ and zero goods with probability $1/4$. This pair of random consumptions is feasible in expectation but is not implementable. To see why, note that in any state of the world in which player 1 is allocated 2 goods, player 2 must be allocated either 1 good, which is never optimal for him, or 0 goods. However, consuming 0 goods with probability $3/4$ is not optimal for player 2.

The utility function of agent 1 and 2 in Example 2 does not satisfy the substitutes property: consider $p^1 = p^2 = 0.5, p^3 = 3$, then the optimal bundle is $\{1, 2\}$; however, when we increase $p^2$ to $q^2 = 3$ and keep other prices unchanged, there is no optimal bundle which includes good 1. Our main results show that examples such as the one above cannot be constructed with utility functions that satisfy the substitutes property.

The assignment of courses to students typically requires a mechanism without transfers, i.e., without a divisible good. To address this and related applications, Theorem 2 demonstrates existence of a competitive equilibrium for the no-transfer economy. Hylland and Zeckhauser (1979) first proposed Walrasian equilibria as an allocation mechanism for the unit demand case. They show that some equilibria may be Pareto inefficient because local non-satiation need not hold in this setting. Nonetheless, Hyland and Zeckhauser (1979) show that efficient equilibria always exist. Mas-Colell (1992) coins the term *strong equilibrium* for a competitive equilibrium in which every consumer chooses the cheapest utility maximizing consumption and shows that strong equilibria are efficient. Our Theorem 2 establishes the existence of a strong and, therefore, Pareto efficient equilibrium.

As illustrated in Example 1, allocation problems often feature constraints on individual or group consumption. In course assignment problems, university rules may constrain students' course selections; in a school choice problem, administrators may restrict parents' choices based on the location of their residence; and, finally, in office allocation problems, choices may be constrained by employee seniority. We analyze such constrained allocation problems in Theorem 3. There, we allow a broad range of constraints on individual consumption and show that our model can incorporate them. In some applications, groups of individuals may face constraints on their joint consumption or there may be aggregate constraint. For example, a university may reserves a certain number of seats in a class for those students who must take this class as a requirement of their majors. In addition, there may be aggregate constraints on the available lab space that constrains the available seats in a number of classes. We analyze such constraints in section 3.

### 1.1 Related Literature

Kelso and Crawford (1982) establish the existence of a Walrasian equilibrium using an ascending tatonnement process. They show that this process converges to a Walrasian equilibrium price vector. Gul and Stacchetti (1999) argue that, in a sense, Kelso and Crawford's substitutes condition is necessary for the existence of equilibrium: that given any utility function that does not satisfies the substitutes property, it is possible to construct an $N$-person economy consisting of an agent with this utility function and $N-1$ agents with substitutes utility functions that has no Walrasian equilibrium.[3] Hence, their result shows that it is impossible to extend Kelso and Crawford's existence result to a larger class of utility functions than those that satisfy the substitutes property.

Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in an unconstrained economy. They circumvent Gul and Stacchetti's impossibility result by imposing joint restrictions on agent's preferences, In particular, they assume that the goods can be partitioned into two sets such that all agents consider good within each element of the partition substitutes and goods in different partition elements complements.

A special class of substitutes preferences are the unit-demand preferences. These describe situations in which agents can consume at most one unit of the divisible good. Leonard (1983) studies unconstrained unit-demand economies and identifies an allocation rule that generalizes the second-price auction and has strong incentive and efficiency properties. His allocation rule is the Walrasian rule together with the lowest equilibrium prices. Hyland and Zeckhauser (1979) are the first to introduce what we have called a no transfer payment unit demand economy. They establish the existence of an efficient Walrasian equilibrium in such economies. Since then, many other papers has extended the idea from Hyland and Zeckhauser (1979) to utilize competitive equilibria as solutions in market design problems. Among them, Ashlagi and Shi (2016) justifies the competitive equilibrium from equal incomes in a market with continuum of agents. Le (2017), He, Miralles, Pycia and Yan (2015) and Echenique, Miralles and Zhang (2018) maintain the assumption of unit-demand preferences, but incorporate general endowment structure, non-EU preferences [4] or priority-based allocations. Mas-Colell (1992) and McLennan (2018) study more general convex economies with productions. There are two major differences between those papers and ours. First, most of them introduce some notion of "slackness" in the definition of Walrasian equilibrium to guarantee its existence, while the equilibrium notion in our paper is standard; Second, they focus on convex (or convexified) economies and thus there is no worry about the implementability problem, but implementation is the key issue in

[3] Yang (2017) finds an error in Gul and Stacchetti's proof and supplies an alternative proof.

[4] Here by non-EU preferences, we mean they allow utility to be nonlinear in probabilities in the convexifed economy.

our setup. We will illustrate the second point further in the discussion for the proof of Theorem 1.

Budish, Che, Kojima and Milgrom (2013) study a variety of probabilistic assignment mechanisms. Our work relates to section 4 of their paper, where they define and show the existence of what they call pseudo-Walrasian equilibrium.[5] In this section, they consider fully separable preferences and establish the existence of efficient pseudo-Walrasian equilibrium. They also describe how individual constraints can be incorporated into pseudo-Walrasian equilibrium.

In Appendix B of their paper, they consider a richer class of preferences adopted from Milgrom (2009). These preferences amount to the closure of unit-demand preferences under satiation and convolution.[6] Ostrovsky and Paes Leme (2015) prove that the closure of unit-demand preferences under endowment and convolution yields a strict subset of substitutes preferences. They identify a rich class of preferences that belong to the latter but not the former. It is easy to check that this class of preferences is also excluded from the class described in Appendix B of Budish, Che, Kojima and Milgrom (2013). Thus, compared to Budish, Che, Kojima and Milgrom (2013), we consider a richer class of preferences and a richer class of constraints. In particular, their analysis of pseudo-Walrasian equilibrium does not include group constraints or aggregate constraints, except for the capacity constraints of each single object.

## 2. The Substitutes Property and the Constrained Economy

Let $H = \{1, \ldots, L\}$ be the set of goods. Subsets of $H$ are consumption bundles.[7] We identify each $A \subset H$ with $x \in X := \{0,1\}^L$ such that $x^j = 1$ if and only if $j \in A$. For any $x \in X$, let $\text{supp}\,(x) = \{k \in H | x^k = 1\}$. A utility on $X$ is a function $u : X \to I\!R \cup \{-\infty\}$. The effective domain of $u$, denoted by $\text{dom}\, u$, is the set $\text{dom}\, u = \{x \in X : -\infty < u(x)\}$. Without loss of generality, we can normalize $u$ so that $u(x) \geq 0$ for all $x \in \text{dom}\, u$. We

[5] Presumably, the qualifier pseudo is to indicate the interjection of fiat money and also to acknowledge various additional constraints that are typically not a part in the definition of a competitive economy. By incorporating these constraints into preferences and technology and by assuming that the mechanism designer/seller values the fiat money, we are able to interpret our equilibria as proper Walrasian equilibria.

[6] See section 2.1 for a discussion of closures under substitutes preserving operations.

[7] We are assuming that there is a single unit of each good. This assumption makes the analysis of the implementability problem easier and is without loss of generality, since we can label each of the multiple units of a good as a distinct good. Equilibrium will ensure that each of these units has the same price.

assume that every agent's overall utility function is quasilinear in the divisible good and, given price vector $p \in I\!R^L_+$, denote with $U_i(x,p) = u(x) - p \cdot x$ the agent's objective function.[8]

For $x, y \in I\!R^L$, we write $x \leq y$ to mean each coordinate of $x$ is no greater than the corresponding coordinate of $y$ and let $x \wedge y$ denote $z \in X$ such that $z^j = \min\{x^j, y^j\}$ for all $j$. Similarly, let $x \vee y$ denote $z \in X$ such that $z^j = \max\{x^j, y^j\}$ for all $j$. Without risk of confusion, we sometimes refer to $u$ as the utility function (instead of saying the utility index associated with the utility function $U$). We let $\chi^j \in X$ denote the good $j$; that is, $\chi^j(k) = 1$ if $k = j$; otherwise, $\chi^j(k) = 0$. Similarly, for any set of indivisible goods $A \subset H$, define $\chi^A \in X$ as follows:

$$\chi^A(k) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{otherwise} \end{cases}$$

Throughout, we will assume that $\operatorname{dom} u \neq \emptyset$ and $u$ is monotone; that is, $x \leq y$ implies $u(x) \leq u(y)$.

### 2.1 The Substitutes Property and the Unconstrained Economy

Define the unconstrained demand correspondence for $u$ as follows:

$$D_u(p) := \{x \in X \mid u(x) - p \cdot z \geq u(y) - p \cdot y \text{ for all } y \in X\}$$

Since $\operatorname{dom} u \neq \emptyset$ and $q \in I\!R^L$, $D_u(p)$ will always lie in the effective domain. The substitutes property states the following: let $x$ be an optimal consumption bundle at prices $p$ and assume that prices increase (weakly) to some $\hat{p}$. Then, the agent must have an optimal bundle at $\hat{p}$ which has her consuming at least as much of every good that did not incur a price increase. The formal definition is as follows:

**Definition:** *The function $u$ has the substitutes property if $x \in D_u(p)$, $p \leq \hat{p}$, $\hat{p}^j = p^j$ for all $j \in A$ implies there exists $y \in D_u(\hat{p})$ such that $y^j \geq x^j$ for all $j \in A$.*

This substitutes property was first introduced by Kelso and Crawford (1982). Since then, numerous alternative characterizations have been identified. For example, the substitutes condition is equivalent to $M^\natural$-concavity: the function $u$ satisfies $M^\natural$*-concavity* if

[8] If the agent has an endowment of non-money goods, the objective function is unchanged since the value of the endowment enters the utility function as a constant.

for all $x, y \in \operatorname{dom} u$, $x^j > y^j$ implies that either $u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$ or there is $k$ such that $y^k > x^k$ such that $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$.[9] Moreover, Gul and Stacchetti (1999) show that that if $u$ satisfies the substitutes property, then it must be submodular[10]

$$u(x) + u(y) \geq u(x \vee y) + u(x \wedge y)$$

Perhaps the best-known subclass of substitutes utility functions are unit demand utilities. These utility functions are appropriate for situations in which each agent can consume at most one unit of the good: $u$ is a *unit demand utility* if

$$u(x) = \max\{u(\chi^j) \,|\, \chi^j \leq x\}$$

There is a class of operations that preserve the substitutes property: let $k > 0$ be an integer, $z \in X$ and $u, v$ be two substitutes utility functions and $\operatorname{dom} u \cap \{x \in X | \sigma(x) \geq k\} \neq \emptyset$. Define,

$$\begin{aligned}
u^z(x) &= u(x \wedge z) \\
u_z(x) &= u(x \vee z) - u(z) \\
w(x) &= \max_{y \leq x}\{u(y) + v(x - y)\} \\
\bar{u}(k, x) &= \max_{\substack{y \leq x \\ \sigma(y) \leq k}} u(y)
\end{aligned}$$

$$\underline{u}(k, x) = \begin{cases} \max_{\substack{y \leq x \\ \sigma(y) \geq k}} u(y) & \text{if } \sigma(x) \geq k \\ -\infty & \text{otherwise.} \end{cases}$$

Call $u^z$ the $z$-constrained $u$, $u_z$ the $z$-endowed $u$, $w$ the convolution (or aggregation) of $u, v$, $\bar{u}(k, \cdot)$ the $k$-satiation of $u$ and $\underline{u}(k, \cdot)$ the $k$-lower bound $u$. It is easy to verify that a $z$-constrained or $z$-endowed $u$ satisfies the substitutes property whenever $u$ does and that the convolution of $u$ and $v$ satisfies the substitutes property whenever $u$ and $v$ both satisfy the substitutes property. Bing, Lehmann and Milgrom (2004) prove that the $k$-satiation of a substitutes utility is a substitutes utility. The following lemma establishes the substitutes property for $k$-lower bound $u$. All proofs can be found in the Appendix.

[9] See, for example, Shioura and Tamura (2015), Theorem 4.1.

[10] Gul and Stacchetti (1999) show this result for a monotone $u$ with $\operatorname{dom} u = H$. Their result extends under the convention that $-\infty \geq -\infty$.

**Lemma 1:** *If $u$ is monotone, satisfies substitutes property and* $\operatorname{dom} u \cap \{x \in X | \sigma(x) \geq k\} \neq \emptyset$*, then* $\underline{u}(k, \cdot)$ *satisfies substitutes property.*

For any given class of utility functions, $\mathcal{U}$, and set of substitutes preserving operations, $\tau$, let $\tau(\mathcal{U})$ denote the set of all utility functions that can be derived from the elements of $\mathcal{U}$ by repeatedly applying various operations in $\tau$. We will call $\tau(\mathcal{U})$ the $\tau$-closure of $\mathcal{U}$. In other words, $\tau(\mathcal{U})$ is the smallest family of utility functions that includes $\mathcal{U}$ and is closed under operations in $\tau$. Clearly, if each element of $\mathcal{U}$ satisfies the substitutes property, then so does the $\tau$-closure of $\mathcal{U}$.

Ostrovsky and Paes Leme (2015) show that the endowment and convolution closure of the set of unit demand preferences is a strict subset of the set of all substitutes preferences.[11] They also provide a rich class of examples that satisfy the substitutes property but are not in the endowment and convolution closure of the set of unit demand preferences. Let $\tau$ be the five substitutes preserving operations discussed above and let $\mathcal{U}$ be the set of all unit demand preferences. Then, using Ostrovsky and Paes Leme's arguments, it is easy to verify that $\tau(\mathcal{U})$ is a strict subset of substitutes preferences and excludes the same rich class of preferences that these authors have identified.

We conclude this section by discussing the existence of equilibrium in an unconstrained economy. Let $N$ be the number of agents in the economy and let $\xi \in X^N$ be an allocation. We write $\xi_i$ for to denote agent $i$'s consumption in the allocation $\xi$. Then $(\xi, p)$ is a (deterministic) Walrasian equilibrium in the unconstrained economy if $\sum_{i=1}^{N} \xi_j \leq \chi^H$, $u_i(\xi_i) - p\xi_i \geq u_i(x) - px$ for all $x \in X$ and all $i$ and $\sum_{i=1}^{N} \xi_j^a = 1$ if $p^a > 0$ for all $a \in H$. Kelso and Crawford (1982) showed that when the preferences satisfy monotonicity and substitutes property, and the effective domain is $X$, then there exists an equilibrium with (deterministic) allocation in the unconstrained economy. To see how the result can be extended to general effective domains, notice that for each $i$ and $p$, as $\operatorname{dom} u_i \neq \emptyset$, the demand $D_{u_i}(p) \subseteq \operatorname{dom} u_i$. Then for any candidate equilibrium allocation $\xi$, it must be the

[11] They conjecture that the endowment and convolution closure of the set of all weighted matroids is the set of substitutes preferences. Ostrovsky and Paes Leme (2015) note that results from Murota (1996), Murota and Shioura (1999), and Fujishige and Yang (2003) ensure that every *weighted matroid* and hence every *rank function* satisfies the substitutes property. For the definitions of weighted matroid, rank function and other relevant terms and results from matroid theory, see Appendix A where we provide a short proof that weighted matroids satisfy the substitutes condition based on Fujishige and Yang (2003)'s result that the substitutes condition is equivalent to $M^\natural$-concavity.

case that $\xi_i \in \operatorname{dom} u_i$ for each $i$. This provides a necessary condition for existence of an equilibrium and we will incorporate it into the definition of an unconstrained economy.

**Definition:** $\mathcal{E}^u = \{(u_i)_{i=1}^N\}$ *is an unconstrained economy if,* $u_i$ *satisfies the substitutes condition for all* $i$ *and there exists an allocation* $\xi$ *such that* $\sum \xi_i \leq \chi^H$ *and* $\xi_i \in \operatorname{dom} u_i$ *for all* $i$.

The following lemma states that the above additional condition is also sufficient for the existence of an equilibrium.

**Lemma 2:** *The unconstrained economy has a (deterministic) equilibrium.*

Notice that efficient allocations of non-money goods, optimal demands and Walrasian equilibria are independent of the initial endowments in the unconstrained economy. Therefore, the definition of the unconstrained economy omits them. However, endowments will matter in the constrained economy, defined in the next section.

### 2.2 The Constrained Economy

When agents have limited budgets, a deterministic equilibrium may not exist even in the simplest constrained economies: assume that there are two agents and a single good. Both agent's utility for the good is 2 but both have only one unit of the divisible good. Without randomization, if the price is less than or equal to 1, both agents will demand the good; if the price is greater than 1 neither will demand the good. Since there is exactly one unit of the good, there can be no equilibrium for this economy. If randomization is allowed, the equilibrium price of the good is 2 and each agent will get the good with probability $1/2$. Thus, we need to extend our definition of an allocation to allow randomness.

A random consumption (of indivisible goods) $\theta : X \to [0, 1]$ is a probability distribution on $X$; that is, $\sum_x \theta(x) = 1$. Let $\Theta$ denote the set of all random consumptions. For $\theta \in \Theta$, let $\bar{\theta} \in I\!R^L_+$ denote the coordinate-by-coordinate mean of $\theta$; that is $\bar{\theta}^j = \sum_x \theta(x) \cdot x^j$. We assume that $u$ is also the agent's von Neumann-Morgenstern utility function. Hence,

$$u(\theta) = \sum_z u(z)\theta(z)$$

The effective domain of $u$ on $\Theta$ consists of all the random consumptions such that $\theta(x) > 0$ implies $x \in \operatorname{dom} u$.[12]

Quasilinearity ensures that we do not have to worry about randomness in the consumption of money; we can identify every such random consumption with its expectation. Hence, we define the von Neumann utility function $U$ as follows:

$$U(\theta, p) = u(\theta) - p \cdot \bar{\theta}$$

Let $w_i \in X$ denote agent $i$'s endowment of indivisible goods and let $b_i$ denote her endowment of money. For some applications, it is useful to have an additional agent, the seller or market designer, who holds some or all of the aggregate endowment of goods. We will sometimes refer to this agent as agent 0 and assume that she derives no utility from goods; she only values money. The aggregate endowment of goods in the economy is $\chi^H := (1, \ldots, 1) \in X$ and, therefore, the seller's endowment of the indivisible goods is $w_0 = \chi^H - \sum_{i=1}^N w_i$. We will assume that $w_i \in \operatorname{dom} u_i$ for each $i$ to guarantee that agent $i$ can afford at least one bundle in the effective domain.

**Definition:** $\mathcal{E} = \{(u_i, w_i, b_i)_{i=1}^N\}$ *is a constrained economy if, for all* $i$, $u_i$ *satisfies the substitutes condition,* $b_i > 0$ *and* $w_i \in \operatorname{dom} u_i$.

A random allocation (of indivisible goods) for this economy is a probability distribution $\alpha : X^N \to [0, 1]$. For any such $\alpha$, let $\alpha_i$ denote the $i$'th marginal of $\alpha$; that is, $\alpha_i \in \Theta$ is the random consumption of agent $i$ with

$$\alpha_i(x) = \sum_{\{\xi : \xi_i = x\}} \alpha(\xi)$$

A random allocation $\alpha$ is feasible for the economy $\mathcal{E}$ if, for all $\xi$ such that $\alpha(\xi) > 0$, $\xi_i \in \operatorname{dom} u_i$ for all $i$ and $\sum_{i=1}^N \xi_i \leq \chi^H$.

The budget set of an agent with endowment $w, b$ at prices $p$ is

$$B(p, w, b) = \left\{ \theta \in \theta \,\middle|\, p \cdot \bar{\theta} \leq p \cdot w + b \right\}$$

[12] The function $u : \Theta \to [0, 1]$ is continuous on the effective domain but not on the whole domain. For example, suppose that $\operatorname{dom} u = X \backslash \{0\}$ and there is a sequence of random consumptions $\theta^n$ such that $\theta^n(0) = 1/n$ and $\theta^n(H) = 1 - 1/n$. Clearly, $u(\theta^n) = -\infty$ for all $n$, but $\theta^n \to \theta$ where $\theta(H) = 1$ and $u(\theta) \in I\!R$.

Then, $\theta \in B(p, w, b)$ is optimal for agent $i$ given the budget $B(p, w, b)$ if

$$U_i(\theta, p) \geq U_i(\theta_o, p)$$

for all $\theta_o \in B(p, w, b)$.

**Definition:** *A price $p \in I\!R^L_+$ and a random allocation $\alpha$ is an equilibrium for the constrained economy $\mathcal{E}$ if*
*(1) $\alpha$ is feasible for the economy $\mathcal{E}$;*
*(2) for all $i$, $\alpha_i$ is optimal for agent $i$ given the budget $B(p, w_i, b_i)$;*
*(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i=1}^N \xi_i^j = 1$.*

**Theorem 1:** *The constrained economy $\mathcal{E} = \{(u_i, w_i, b_i)_{i=1}^N\}$ has an equilibrium.*

One possible way to prove Theorem 1 would be prove existence for the *convexified economy*; that is, an economy in which agents have convex consumption sets. To see how this can be done, let $Z = [0, 1]^L$ and define, $v : Z \to I\!R$ the *convexified version* of $u$ as follows:

$$v(\zeta) = \max\{u(\theta) \,|\, \bar{\theta} \leq \zeta\}$$

and define $V(\zeta, p) = v(\zeta) - p \cdot \zeta$. Hence, $v(\zeta)$ is the maximum utility that the agent with von Neumann utility index $u$ could get by choosing a lottery $\theta$ over indivisible goods such that the expected consumption of good $i$ with $\theta$ is no greater than $\zeta$. Note that $V$ is defined on the convex set $Z \times I\!R^L_+$. An allocation $(\zeta_1, \ldots, \zeta_N)$ is feasible in the convexified economy if $\sum \zeta_i \leq \chi^H$. Thus, the convexified economy ignores the implementability part of feasibility.

Establishing the existence of equilibrium in the convexified economy using standard techniques is straightforward. [13] The last step for this method of proof would be to find a way to implement the equilibrium of the convexified economy with a random allocation. This last step is not difficult for a unit demand economy since, for that case, an appeal to the Birkhoff-von Neumann Theorem ensures existence of the desired random allocation.

[13] For example, it can be shown that the convexified version of utility function satisfies the conditions in McLennan (2018) and thus existence of equilibrium in the convexified economy can be always guaranteed for all utility index $\{u_i\}_{i=1}^N$. However, as is shown in the Example 2, without substitutes property, there might be no equilibrium in the economy even when there is one in its convexified version.

This line of argument does not work generally when agents can consume multiple units of indivisible goods and preferences are not separable. Example 2 in the introduction shows that, without the substitutes property, the implementability problem is, in general, insurmountable. Even with the substitutes property, Example 1 in the introduction shows that not all equilibrium consumption plans of the convexified economy are implementable.

Our proof relies on the existence of equilibrium in the unconstrained economy. We seek a $\lambda_i \in (0, 1]$ for each agent $i$ and a Walrasian equilibrium $(p, \alpha)$ for the modified *unconstrained* economy with random consumption in which each $u_i$ is replaced by

$$\hat{u}_i = \lambda_i u_i$$

such that each agent $i$ spends, in expectation, (1) no more that $b_i$ on indivisible goods and (2) exactly $b_i$ on indivisible goods if $\lambda_i < 1$. It is possible to decrease an agent's equilibrium spending as much as needed by decreasing that agent's $\lambda_i$. Hence, we can satisfy condition (1). A fixed-point argument ensures that we can also satisfy condition (2). The Walrasian equilibria for this modified economy are then shown to be equilibria of the original economy.

## 3. No Transfer Payments Economy and Constraints

In this section, we will consider allocation problems in settings without a divisible good. We call this type of an economy a *no transfer payments economy.* In many applications, no transfer payments economies impose constraints on individual consumption, or on the consumption of groups. To address some of these applications, we show how our model can incorporate a variety of individual, group, and aggregate constraints. An individual constraint restricts the number of goods that a single agent can consume from a specified set of goods. A group constraint restricts the total number of goods that can be consumed from a specified set of perfect substitutes by a particular group. Finally, aggregate constraints restrict the various combinations of goods available for the entire population.

An example of an individual constraint is Princeton University's *rule of 12.* According to this rule, no more than 12 courses in a student's major may be counted towards the 31 courses needed to obtain the A.B. degree. Minimal distribution requirements are a second type of individual constraints. For example, Art and Archaeology students at Princeton University must take at least one course in each of the following three areas: group 1 (ancient), group 2 (medieval/early modern), and group 3 (modern/contemporary). An example of a group constraint is the requirement that at least 50 percent of the slots in each school should go to students who live in the school's district. Similarly, the so-called "controlled choice" constraints in school assignment that require schools to balance the gender, ethnicity, income, and test score distributions among their students, are group constraints.[14] Aggregate constraints may reflect aggregate feasibility constraints. For example, suppose two versions of introductory physics are being offered: Phy 101, the version that does not require calculus and Phy 103, the version that does require calculus. Suppose each of these classes can accommodate 120 students. However, because both courses have lab requirements and lab facilities are limited, the total enrollment in the two courses can be no greater than 200 students.

In the next subsection, we describe the no transfer payments economy, define a strong (Walrasian) equilibrium and establish its existence and efficiency. Section 3.2 deals with individual, group and aggregate constraints.

### 3.1 No Transfer Payments Economy

In a no transfer payments economy, each agent $i$ has a substitutes utility function $u_i$ and a quantity $b_i$ of fiat (or artificial) money. Initially, the entire aggregate endowment belongs to the market designer. Each agent's utility depends only on her consumption of indivisible goods. Hence, agents solve the following utility maximization problem:

$$\max u_i(\theta) \text{ subject to } p \cdot \bar{\theta} \leq b_i$$

Let $M_i(p, b_i)$ be the solutions to the above maximization problem.

**Definition:** $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ *is a no transfers payments economy if, for all* $i$, $u_i$ *satisfies the substitutes condition,* $0 \in \mathit{dom}\, u_i$ *and* $b_i > 0$.

[14] See Abdulkadiroğlu, Pathak and Roth (2005) for examples of such constraints in practice.

In the no transfer payments setting, Walrasian mechanisms provide a rich menu of allocation mechanisms with desirable properties. The designer may accommodate fairness concerns by choosing the agents' endowments of fiat money (the $b_i$'s) appropriately. In particular, choosing the same $b_i$ for every agent ensures that the resulting allocations are envy-free. This is the setting for many allocations problems such as class selection, school choice, or office selection when a business or a department moves into a new building. In such markets, the Walras correspondence can serve both as real allocation mechanism and as a benchmark for evaluating other mechanisms.

Hyland and Zeckhauser (1979) note that in a no transfer payments economy with unit-demand preferences, some Walrasian equilibria are inefficient. Specifically, no transfer payments economies may have equilibria in which some agents do not purchase the least expensive optimal option in their budget sets and equilibria with this property may be inefficient. To address this problem, MasColell (1992) introduces the concept of a *strong* equilibrium, that is, a Walrasian equilibrium in which every consumer chooses the least expensive optimal bundle and proves that strong equilibria are Pareto efficient.

**Definition:** *A price $p \in I\!R^L_+$ and a random allocation $\alpha$ is a strong equilibrium for the no transfers payments economy $\mathcal{E}^*$ if*

*(1) $\alpha$ is feasible for the economy $\mathcal{E}^*$;*

*(2) for all $i$, $\alpha_i \in M_i(p, b_i)$ and $p\alpha_i \leq p\theta$ for all $\theta \in M_i(p, b_i)$;*

*(3) $p^j > 0$ and $\alpha(\xi) > 0$ imply $\sum_{i \geq 1} \xi_i^j = 1$.*

The theorem below establishes the existence of a strong and, therefore, Pareto efficient equilibrium for the no transfer payments economy.

**Theorem 2:** *The no transfer payments economy has a strong equilibrium.*

Our proof of Theorem 2 relies on Theorem 1: we consider the sequence of constrained economies $\mathcal{E}_n = \{(nu_i, w_i, b_i)_{i=1}^k\}$ for $n = 1, 2 \ldots$ where $w_i^j = 0$ for all $j$ and $i$. Hence, $\mathcal{E}_n$ is a constrained economy in which agent $i$'s endowment of goods is equal to her endowment of goods in $\mathcal{E}^*$ (i.e., zero), her endowment of money is the same as her endowment of fiat money in $\mathcal{E}^*$ and her utility function is $n$-times her utility function in $\mathcal{E}^*$. Then, we appeal to Theorem 1 to conclude that each $\mathcal{E}_n$ has an equilibrium $(p^n, \alpha^n)$. Since this sequence lies in a compact set, it has a limit point. Then, we show that this limit point must be an equilibrium of $\mathcal{E}^*$. Since this equilibrium is a limit-point of a sequence of equilibria for constrained economies; that is, equilibria in which money has intrinsic value, it must be a strong equilibrium.

### 3.2 Group Constraints

In many applications, one group is given priority over another. For example, suppose that the maximal enrollment in a particular physics class is $n$ and there are $m < n$ physics majors who are required to take that class. Thus, at most $n - m$ non-majors can enroll in the class. More generally, a group constraint $(A, n)$ for the group $I \subset \{1, \dots, N\}$ states that the agents in $I$ can collectively consume at most $n$ units from the set $A$, where $A$ is a collection of *perfect substitutes* (for all agents).

To accommodate this constraint, pick any $n$ element subset $B$ of $A$ and let $C = (A \backslash B)^c$. Then, replace each $u_i$ for $i \in I$ with $u_i'$ such that

$$u_i'(x) = u_i\left(x \wedge \chi^C\right)$$

Thus, the new utility for members of group $I$ is the their original utility restricted to elements of $C$ (that is, excluding elements in $A \backslash B$). As we noted above, restrictions of utilities to a subset of choices satisfy the substitutes condition if the original utility satisfies the substitutes condition. Moreover, since elements of $A$ are perfect substitutes, restricting members of group $I$ to $C$ is equivalent to restricting their aggregate consumption of the good represented by the elements of $A$. Thus, a group constraint can be accommodated by modifying some agents' utility function.

### 3.3 Individual Constraints

The simplest individual constraints are bounds on the number of goods an agent may consume. For example, a student may be required to take 4 classes each semester, but may not enroll in more than 6. We can incorporate this constraint by modifying the student's utility function without constraints $u$ as follows:

$$u^m(x) = \max_{\substack{y \leq x \\ \sigma(y) \leq 6}} \hat{u}(y)$$

where

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } \sigma(x) \geq 4 \\ -\infty & \text{if } \sigma(x) < 4 \end{cases}$$

The modified utility $u^m$ incorporates the lower bound constraint by adjusting the effective domain of $u$ to those bundles that satisfy the constraint. It incorporates the upper bound constraint by imposing satiation above the constraint. Next, we generalize these constraints and impose bounds on overlapping subsets of goods. To preserve the gross substitutes condition, we require that the utility be separable across subsets of goods that must satisfy a constraint. A collection of goods, $A \subset H$, is a *module* for the utility $u$ if

$$u(x) = u(x \wedge \chi^A) + u(x \wedge \chi^{A^c})$$

where $A^c$ is the complement of $A$ in $H$. Note that this condition is symmetric: if $A$ is a module, then so is $A^c$. For example, suppose that $A$ is the set of all humanities courses and $A^c$ is the set of all science courses. If a student's utility for various combination of science courses is independent of her utility over various combinations of humanities courses, then $A$ is a module. A collection of sets, $\mathcal{H}$, is a *hierarchy* if $A, B \in \mathcal{H}$ and $A \cap B \neq \emptyset$ implies $A \subset B$ or $B \subset A$. Given any $u$, we say that the hierarchy $\mathcal{H}$ is *modular* if each element of $\mathcal{H}$ is a module of $u$.

A modular constraint places bounds on the agent's consumption for subsets of items that form a modular hierarchy. The collection $c = \{(A(k), (l(k), h(k))_{k=1}^K\}$ is a *constraint* if $l(k), h(k)$ are integers, $A(k) \subset H$, and $X_c \cap X^c \neq \emptyset$ where

$$X_c := \{x \in X | \, \forall k, \sigma(x \wedge \chi^{A(k)}) \geq l(k)\}$$

are consumptions that satisfy the lower bound and

$$X^c = \{x \in X | \, \forall k, \sigma(x \wedge \chi^{A(k)}) \leq h(k)\}$$

are consumptions that satisfy the upper bound. The constraint $c = \{(A(k), (l(k), h(k))_{k=1}^K\}$ is a *modular constraint* for $u$ if $\mathcal{H} = \{A(1), \dots, A(K)\}$ is a modular hierarchy for $u$ and $X_c \cap \operatorname{dom} u \neq \emptyset$.

As an example of modular constraint, suppose that students must take at least 3 humanities classes and at least 4 social science classes; moreover, each student is required to take at least 8 but no more than 12 classes overall. In this case, the constraint is modular if the student's utility over combinations of science courses is independent of her utility over combinations of humanities courses.

Given a utility $u$, let $\hat{u}_c$ be the utility function with effective domain $X_c$, that is,

$$\hat{u}_c(y) = \begin{cases} u(y) & \text{if } y \in X_c \\ -\infty & \text{otherwise.} \end{cases}$$

Finally, define $u(c, \cdot)$ be the following utility function:

$$u(c, x) = \max \left\{ \hat{u}_c(y) \,\middle|\, y \in X^c, y \leq x \right\}$$

Then the effective domain of $u(c, \cdot)$ is $\operatorname{dom} u(c, \cdot) = X_c \cap \operatorname{dom} u \neq \emptyset$.

**Lemma 3:** *Let $u$ satisfy the substitutes condition. Then, if $c$ is a modular constraint set for $u$, $u(c, \cdot)$ satisfies the substitutes property.*

To illustrate how the substitutes property may fail if the constraints are not modular, consider the utility function described in equation (1) below. Let $H = \{0, 1, 2, 3\}$. Then,

$$u(x) = \begin{cases} 2 & \text{if } x^j \cdot x^{j \oplus 1} > 0 \text{ for some } j \in H \\ 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

where $\oplus$ denotes addition modulo 4. Note that $u$ satisfies the substitutes condition.[15] Let $A = \{0, 1\}$ and suppose that the agent is constrained to consuming at most on unit from $A$. To see that the resulting constrained utility function does not satisfy the substitutes property, set $p^0 = p^1 = p^2 = p^3 = 1/2$. Then, $\{0, 3\}$ is an optimal consumption set at prices $p$. The substitutes property fails since at prices $q$ such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$ there is no optimal bundle that contains item 0.

To illustrate how the substitutes property may fail if the modular constraints do not form a hierarchy, consider the separable utility function where $u(A) = |A|$. Let $H = \{0, 1, 2, 3\}$, then any subset of $H$ is a module of $u$. Suppose the constraints are $(\{1, 2\}, 0, 1)$, $(\{0, 1\}, 0, 1)$ and $(\{0, 1, 2, 3\}, 0, 2)$. Then, at $p^j = 1/2$ for all $j \in H$, then $\{1, 3\}$ is an optimal

[15] It is a convolution of the two unit demand preferences $v$ and $\hat{v}$ where $v$ takes the value 1 at any $x$ such that $x^0 > 0$ or $x^2 > 0$ and is equal to zero otherwise and $\hat{v}$ takes the value 1 at any $x$ such that $x^1 > 0$ or $x^3 > 0$ and is equal to zero otherwise.

consumption set at $p$. Again, the substitutes property fails since at prices $q$ such that $q^3 = 2$ and $q^j = p^j$ for $j \neq 3$ there is no optimal consumption set that contains 1.

To establish existence of equilibrium we must guarantee that the interior of each agent's budget set contains some consumption in the effective domain of her modified utility. To address this issue, we add two assumptions to our earlier model. First, we assume that there exists a division of the aggregate resources into $N + 1$ consumptions such that every element meets every consumer $i$'s lower bound constraint. Second, we assume that all agents have equal endowments of fiat money, which can be normalized to 1.

**Definition:** $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$ *is a no transfers payments economy with modular constraints if,*

*(i) for all $i$, $u_i$ satisfies the substitutes property;*

*(ii) for all $i$, $c_i$ is a modular constraint for $u_i$;*

*(iii) there is $x_1, \cdots, x_{N+1}$ such that $\sum_{k=1}^{N+1} x_k \leq \chi^H$ and $x_k \in \operatorname{dom} u_i(c_i, \cdot)$ for all $k, i$.*

**Theorem 3:** *The no transfer payments economy with modular constraints has a strong equilibrium.*

The role of (iii) and equal money endowments is to ensure that every agent can afford a consumption in the effective domain of her utility function. Alternatively, we could assume that there is a subset of goods in abundant supply (goods that have zero price in equilibrium) and agents can choose a consumption in the effective domain from that subset. In our course selection application, it may be the case that a subset of classes is never oversubscribed and students can choose courses that meet the requirements from that subset. In that case, item (iii) in the definition above and the assumption of equal budgets could be dispensed with.

### 3.4 Aggregate Constraints

The above described example of two physics classes that must obey a constraint on available lab space is an aggregate constraint. Alternatively, suppose an economics department schedules classes in labor economics, intermediate microeconomics and in corporate finance. There are two types of TAs, those that can cover labor economics and microeconomics, and those that can cover microeconomics and corporate finance. TA time is fungible across different classes so that at most 60 students can be enrolled labor economics, at most 60 students can be enrolled in corporate finance and at most 120 students can be enrolled in any of the three types of classes.

In these examples, we can describe the aggregate constraint as a hierarchy $\mathcal{H}$ that constrains the supply of available items. That is, the aggregate constraint has the form $c = \{(A(k), n(k))_{k=1}^K\}$ such that the $A(k) \subseteq H$ for all $k$, $\{A(k)\}_{k=1}^K$ constitute a hierarchy and each $n(k)$ is a natural number describing the maximal number of goods that can be supplied from set $A(k)$.

Hierarchies are a special case of a class of constraints that can be described through matroids. A nonempty, finite collection of sets $\mathcal{B} \subset X$ is a *basis system* if $x, y \in \mathcal{B}$ and $x^j > y^j$ implies there exists $k$ such that $y^k > x^k$ and $x - \chi^j + \chi^k \in \mathcal{B}$.[16] Let $\mathcal{B}$ be the set of all feasible output combinations; that is, $\mathcal{B}$ is the technology for a no transfer payments economy. We say that the technology $\mathcal{B}$ is *matroidal* if $\mathcal{B}$ is a basis system. A no transfer payments production economy is a collection $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ such that $|\mathcal{B}| > 1$. We exclude the case in which the production set has a single element because this case corresponds to the standard case without production covered in Theorem 2.

To see how we can embed a collection of aggregate restrictions into a matroidal technology, let $(A, n)$ denote a single aggregate restriction. Hence, the set of feasible production plans given any $X$ and the restriction $(A, n)$ is:

$$X(A, n) = \{x \in X \,|\, \sigma(x \wedge \chi^A) \leq n\}$$

As in the case of individual and group constraints, we can impose multiple nested aggregate constraints: that is, a hierarchy of aggregate constraints. Given any hierarchy of aggregate restrictions $d = \{(A(k), n(k))_{k=1}^K\}$, let $\mathcal{I}_d$ denote the set of all production plans consistent with $d$; that is,

$$\mathcal{I}_d = \bigcap_{a \in d} X(a)$$
$$\mathcal{B}_d = \{x \in \mathcal{I}_d \,|\, \sigma(x) \geq \sigma(y) \text{ for all } y \in \mathcal{I}_d\}$$

[16] A basis system is one of many ways to describe a matroid. See Appendix A, for the definition of a matroid and a few other relevant notions, results from matroid theory.

Hence, elements of $\mathcal{B}_d$ are the vectors in $I_d$ with the largest cardinality of support. Although the natural production set should be $\mathcal{I}_d$, it is without loss of generality to focus solely on the basis system $\mathcal{B}_d$, since agents' preferences are monotone, in equilibrium prices must be weakly positive and profit maximizing will be achieved by maximal production plans in $\mathcal{I}_d$, that is, by elements in $\mathcal{B}_d$.

**Lemma 4:** *If $d$ is a hierarchical collection of aggregate constraints and $\mathcal{B}_d \neq \emptyset$, then $\mathcal{B}_d$ is matroidal.*

In the economy with production, a random allocation $\alpha$ is a probability distribution over $X^N \times \mathcal{B}$. For any such $\alpha$, the marginal $\alpha_i$ is the random consumption for agent $i = 1, \ldots, N$ and the marginal $\alpha_{N+1}$ is the production plan for the producer or seller.

**Definition:** *$\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ is a production economy with no transfer payments if $\mathcal{B}$ is matroidal and if, for all $i$, $u_i$ satisfies the substitutes property and $0 \in \operatorname{dom} u_i$.*

A random allocation $\alpha$ is *feasible* for the economy $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ if, for all $(\xi, z)$ such that $\alpha(\xi, z) > 0$, $\sum_{i=1}^N \xi_i \leq z$, $z \in \mathcal{B}$ and $\xi_i \in \operatorname{dom} u_i$ for all $i$. The definitions of budget sets and consumer optimality remain unchanged.

**Definition:** *A price $p \in I\!R_+^L$ and a random allocation $\alpha$ is a strong equilibrium for the production economy with no transfer payments $\tilde{\mathcal{E}}$ if*

*(1) $\alpha$ is feasible for $\tilde{\mathcal{E}}$;*

*(2) for all $i$, $\alpha_i \in M_i(p, b_i)$ and $p\alpha_i \leq p\theta$ for all $\theta \in M_i(p, b_i)$;*

*(3) $pz \geq pz'$ for all $z' \in \mathcal{B}$ and all $z$ such that $\sum_{\xi \in X^N} \alpha(\xi, z) > 0$.*

*(4) $p^j > 0$ and $\alpha(\xi, z) > 0$ imply $\sum_{i=1}^N \xi_i^j = z^j$.*

Hence, with production, a Walrasian equilibrium specifies prices, a random allocation and a random production plan. The implied random consumption and production plans must be feasible and optimal for both the consumers and the producer. The definition of a strong equilibrium is as in the previous section: the Walrasian equilibrium $(p, \alpha)$ is a strong equilibrium if for each agent $i$, $\alpha_i$ is the cheapest optimal random consumption for $i$ given the budget constraint.

**Theorem 4:** *A strong equilibrium for production economy $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ exists if $\mathcal{B}$ is a matroidal technology. Strong equilibrium allocations are Pareto efficient.*

For the more general statement of type of constraints that can be incorporated in the no transfer payments economies, we can combine the last two theorems.

**Definition:** $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{B}\}$ *is a production economy with no transfers payments and modular constraints if*

*(1)* $u_i$ *satisfies the substitutes property for all* $i$*;*

*(2)* $c_i$ *is a modular constraint for* $u_i$ *for all* $i$*;*

*(3) for all* $z \in \mathcal{B}$*, there is* $x_1, \cdots, x_{N+1}$ *such that* $\sum_{k=1}^{N+1} x_k \leq z$ *and* $x_k \in dom\, u_i(c_i, \cdot)$ *for all* $k, i$*.*

**Corollary:** *The production economy with no transfer payments and modular constraints has a strong equilibrium.*

Part (3) of the definition above requires that every production plan $z \in \mathcal{B}$ can be divided into $N+1$ consumptions in a way that each satisfies every lower bound constraint. This mirrors a similar assumption in Theorem 3 above.

## 4. Conclusion

Our results suggest that Walrasian methods can be employed in a variety of market design problems whenever preferences satisfy the substitutes condition. Gul and Stacchetti (1999) show that given any utility function that does not satisfy the substitutes condition, it is possible to construct an unconstrained economy with $N$ agents, one with the preference in question and $N-1$ with a substitutes preference such that no equilibrium exists. Hence, it seems unlikely that an existence result for the no transfer payments economy that permits a larger set of preferences than the substitutes class can be proved.

However, Sun and Yang (2006) provide a generalization of the Kelso-Crawford existence result that allows for some complementarities in consumption. In particular, they show that if the goods can be partitioned into two sets such that all agents considers good within each element of the partition as substitutes and consider goods in different elements as complements, then a Walrasian equilibrium exists in the corresponding unconstrained economy. A further generalization of this result is offered in Shioura and Yang (2015). One possible extension of the current work would be the see if equilibrium also exists with Sun-Yang preferences in the constrained and no transfer payments economies.

## 5. Appendix A

Unless indicated otherwise, the definitions and results below can be found in Oxley (2011):

A *matroid* $\mathcal{I} \subset X$ is a collection of sets such that (I1) $\emptyset \in \mathcal{I}$, (I2) $y \in \mathcal{I}$, $x \leq y$ implies $x \in \mathcal{I}$ and (I3) $x, y \in \mathcal{I}$, $\sigma(x) < \sigma(y)$ implies there is $j$ such that $x^j < y^j$ and $x + \chi^j \in \mathcal{I}$.

There are various alternative ways to describe a matroid. One way is by characterizing it maximal elements. For any matroid $\mathcal{I}$, let $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \geq x$ and $y \in \mathcal{I}$ implies $y = x\}$ be the set of all maximal elements of $\mathcal{I}$. Then, $\mathcal{B}(\mathcal{I})$ is a *basis system*; that is, (B1) $\mathcal{B}(\mathcal{I})$ is nonempty and (B2) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is $k$ such that $y^k > x^k$ and $x - \chi^j + \chi^k \in \mathcal{B}(\mathcal{I})$.

If $\mathcal{B} \subset X$ satisfies (B1) and (B2), then $\mathcal{I} = \{x \in X \,|\, x \leq y$ for some $y \in \mathcal{B}\}$ is a matroid and $\mathcal{B} = \mathcal{B}(\mathcal{I})$. Every basis system $\mathcal{B}$ satisfies the following stronger version of (B2): (B2$^*$) $x, y \in \mathcal{B}(\mathcal{I})$ and $x^j > y^j$ implies there is $k$ such that $y^k > x^k$ and $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in \mathcal{B}(\mathcal{I})$. Also, all elements of a basis system have the same cardinality; that is, if $x, y \in \mathcal{B}$ and $\mathcal{B}$ is a basis system, then $\sigma(x) = \sigma(y)$. Hence, for any matroid $\mathcal{I}$, $\mathcal{B}(I)$ is the set of elements of $\mathcal{I}$ with the maximal cardinality; $\mathcal{B}(\mathcal{I}) = \{x \in \mathcal{I} \,|\, y \in \mathcal{I}$ implies $\sigma(x) \geq \sigma(y)\}$.

Gul and Stacchetti (2000) show that if $u$ satisfies the substitutes condition, then the set of elements of $D_u(p)$ with the smallest cardinality is a basis system for every $p$.

For any $\mathcal{B}$, let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. If $\mathcal{B}$ is a basis system, then $\mathcal{B}^\perp$ is also a basis system and is called the *dual* of $\mathcal{B}$.

A function $r : X \to I\!N$ is a *rank function* if (R1) $0 \leq r(x) \leq \sigma(x)$, (R2) $x \leq y$ implies $r(x) \leq r(y)$ and (R3) $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$. For any rank function, $r$, the set of all minimal (in the natural order on $I\!R^L$) maximizers of $r$ is a basis system. Also, given any matroid $\mathcal{I}$, the function $r$ defined by $r(x) = \max\{\sigma(y) \,|\, y \leq x, y \in \mathcal{I}\}$ is a rank function.

A *weighted matroid* is a function $\rho$, defined as follows: for any collection of weights $\gamma^j \geq 0$ for $j \in H$ and a matroid $\mathcal{I}$, let $\rho(x) = \max_{\substack{y \leq x \\ y \in \mathcal{I}}} \sum_j \gamma^j \cdot y^j$.

**Fact:** Every weighted matroid satisfies the substitutes condition.

**Proof:** Murota (2009) shows that a weighted matroid is $M^\sharp$ concave. Since $M^\sharp$ concavity is equivalent to the substitute condition, the fact follows. □

Since a rank function is a special case of a weighted matroid, one in which all of the $\gamma^j$'s are equal to 1, it too satisfies the substitutes property.

Recall that in Example 1, the utility function $u$ of students is similar to a weighted matroid except that the effective domain is no longer than $X$. Notice that if we denote the set of consumption bundles which satisfy the constraints and contain 3 classes as $\mathcal{B}$, then $\mathcal{B}$ is a basis system and the effective domain is $\operatorname{dom} u = \{x \in X : \exists y \in \mathcal{B}, s.t. x \geq y\}$. The following lemma states that weighted matroids with such effective domains satisfy substitutes property.

**Lemma A1:** *For any $\gamma^j \geq 0$ for $j \in H$ and a basis system $\mathcal{B}$, denote $\mathcal{A} = \{x \in X : \exists y \in \mathcal{B}, s.t. x \geq y\}$ and*

$$v(\mathcal{A}, x) = \begin{cases} \max_{\substack{y \leq x \\ y \in \mathcal{A}}} \sum_j \gamma^j \cdot y^j & \textit{if } x \in \mathcal{A} \\ -\infty & \textit{otherwise.} \end{cases}$$

*then $v$ satisfies substitutes property.*

**Proof:** We will directly check that $v$ satisfies $M^\sharp$-concavity. Suppose that $x, y \in \operatorname{dom} u = \mathcal{A}$ and $x^j > y^j$. By definition of $\mathcal{A}$, there exists $\hat{x}, \hat{y} \in \mathcal{B}$ such that $\hat{x} \leq x$, $\hat{y} \leq y$. If $\hat{x}^j = 0$, then $x - \chi^j \in \mathcal{A}$ as $\hat{x} \leq x - \chi^j$ and $y + \chi^j \in \mathcal{A}$. Thus $v(x - \chi^j) + v(y + \chi^j) = \sum_i \gamma^i \cdot (y^i + x^i) = v(x) + v(y)$. If instead $\hat{x}^j = 1$, then $\hat{x}^j > \hat{y}^j$ and by the definition of a basis system, there exists $k$ such that $\hat{y}^k > \hat{x}^k$ and $\hat{x} - \chi^j + \chi^k, \hat{y} - \chi^k + \chi^j \in \mathcal{B}$. Then $x - \chi^j + \chi^k, y - \chi^k + \chi^j \in \mathcal{A}$ and $v(x - \chi^j + \chi^k) + v(y + \chi^j - \chi^k) = \sum_i \gamma^i \cdot (y^i + x^i) = v(x) + v(y)$. This implies v satisfies $M^\sharp$-concavity and thus substitutes property. □

If we set $\gamma^j = 2$ for all $j \in C^*$, $\gamma^j = 0$ for all $j \in C$ and $\mathcal{B}$ as the set of consumption bundles which satisfy the constraints and contain 3 classes, then the utility function of students in Example 1 can be rewritten as in the above lemma and thus it satisfies substitutes property.

# 6. Appendix B

## 6.1 Proof of Lemma 1

To begin with, we need to modify the definition of single improvement property (SI) (Gul and Stacchetti, 1999) as follows:

**Definition:** *The function $u$ has the single improvement property (SI) if for each $p$ and $x \notin D_u(p)$, then either $x \notin \operatorname{dom} u$ or there exists $y$ such that $U(x,p) < U(y,p)$, $\#(supp\,(x) - supp\,(y)) \leq 1$ and $\#(supp\,(y) - supp\,(x)) \leq 1$.*

By Theorem 4.1 and Theorem 5.1 in Shioura and Tamura (2015), we know that substitutes property, (SI) and $M^\natural$-concavity are equivalent. Also, a utility function $u$ is submodular if it satisfies the substitutes property. Gul and Stacchetti (1999) show that (SI) is equivalent to the substitutes property for the effective domain $X$. The above modified definition is equivalent to the substitutes property for a general effective domain.

The following proof is similar to the proof that $k$-satiation preserves substitutes property in Bing, Lehmann and Milgrom (2004). We first prove two auxiliary lemmas; Lemma B1 strengthens $M^\natural$-concavity for the sub-case where there is no ranking of the two chosen bundles.

**Lemma B1:** *Let $u$ be a monotone utility that satisfies the substitutes property. If $x, y \in \operatorname{dom} u$ with $x \ngeq y$ and $y \ngeq x$, then there is $k, l$ such that $x^j > y^j$, $y^k > x^k$ such that $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$.*

**Proof:** Since $u$ satisfies substitutes property, $u$ is $M^\natural$ concave. Since $y \ngeq x$ it follows that there exists $j$ with $x^j > y^j$. Since $x, y \in \operatorname{dom} u$ the definition of $M^\natural$-concavity implies that either there is $k$ such that $y^k > x^k$ such that $u(x - \chi^j + \chi^k) + u(y + \chi^j - \chi^k) \geq u(x) + u(y)$ and we are done, or that $u(x - \chi^j) + u(y + \chi^j) \geq u(x) + u(y)$. That is, $u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$. Since $x \ngeq y$, there exists $l$ with $y^l > x^l$. By $M^\natural$-concavity it follows that either there is $j_0$ with $x^{j_0} > y^{j_0}$ such that $u(x - \chi^{j_0} + \chi^l) + u(y + \chi^{j_0} - \chi^l) \geq u(x) + u(y)$ and we are done, or

$$u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

Since $u$ is submodular, it follows that, for any $x \leq y$, $x^k = 1$ and $x \in \operatorname{dom} u$,

$$u(y) - u(y - \chi^k) \leq u(x) - u(x - \chi^k)$$

The above two inequalities together with the fact that $x - \chi^j + \chi^l, y - \chi^l + \chi^j \in \operatorname{dom} u$ since $x, y \in \operatorname{dom} u$ and $u$ is monotone. Thus, we have:

$$u(x - \chi^j + \chi^l) - u(x - \chi^j) \geq u(x + \chi^l) - u(x) \geq u(y) - u(y - \chi^l)$$

and

$$u(y - \chi^l + \chi^j) - u(y - \chi^l) \geq u(y + \chi^j) - u(y) \geq u(x) - u(x - \chi^j)$$

Adding the above two inequalities we obtain

$$u(x - \chi^j + \chi^l) + u(y - \chi^l + \chi^j) \geq u(x) + u(y)$$

as desired. □

The following Lemma states that, if a bundle of $n$ elements is not optimal for bundles with at least $n$ elements, then we can either add an element or change an element to increase its utility.

**Lemma B2:** *Let $u$ be a monotone utility that satisfies the substitutes property. Let $A, B$ be such that $\#(B) \geq n = \#(A)$, $\chi^A \in \operatorname{dom} u$ and $U(\chi^B, p) > U(\chi^A, p)$. Then, either there exists $l \notin A$ such that $U(\chi^A + \chi^l, p) > U(\chi^A, p)$ or there exists $k \in A$, $l \notin A$ such that $U(\chi^A + \chi^l - \chi^k, p) > U(\chi^A, p)$.*

**Proof:** Denote $\chi^B$ as the optimal consumption bundle at price $p$ among all bundles with at least $n$ elements. Among those $B$, take $B^*$ as the set that minimizes the Hausdorff distance from $A$: $d(A, B) = \#(A - B) + \#(B - A)$. By assumption, $U(\chi^{B^*}, p) > U(\chi^A, p)$. Clearly $B^* - A \neq \emptyset$, otherwise, by $\#(A) = n, \#(B^*) \geq n$ and $B^* \subseteq A$, $A = B^*$, a contradiction. Since $x^A \in \operatorname{dom} u$, we have $x^{B^*} \in \operatorname{dom} u$.

First, assume $A - B^* \neq \emptyset$. By Lemma B1, there exists $k, l$ with $k \in A - B^*$, $l \in B^* - A$ such that

$$u(\chi^A) + u(\chi^{B^*}) \leq u(\chi^A - \chi^k + \chi^l) + u(\chi^{B^*} - \chi^l + \chi^k)$$

Notice that the total costs of bundles on both sides of the above inequality is the same. Therefore, we obtain the following relationship for net utilities:

$$U(\chi^A, p) + U(\chi^{B^*}, p) \leq U(\chi^A - \chi^k + \chi^l, p) + U(\chi^{B^*} - \chi^l + \chi^k, p)$$

By definition, $U(\chi^{B^*}, p) > U(\chi^A, p)$, $U(\chi^{B^*}, p) \geq U(\chi^A - \chi^k + \chi^l, p)$ and $U(\chi^{B^*}, p) \geq U(\chi^{B^*} - \chi^l + \chi^k, p)$. If $U(\chi^{B^*}, p) = U(\chi^{B^*} - \chi^l + \chi^k, p)$, then $(\chi^{B^*} - \chi^l + \chi^k)$ is also optimal at price $p$ among all bundles with at least $n$ elements. Moreover, $d(A, B^* \cup \{k\} - \{l\}) < d(A, B^*)$, which contradicts with the definition of $B^*$. Thus, $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^l + \chi^k, p)$ and by the inequality, $U(\chi^A, p) < U(\chi^A - \chi^k + \chi^l, p)$ and we are done.

Second, assume $A - B^* = \emptyset$. As $A \neq B^*$, $A$ is a strict subset of $B^*$ and $\#(B^*) \geq n+1$. For any $j \in B^* - A$, $d(B^* - \{j\}, A) < d(B^*, A)$ and $\#(B^* - \{j\}) \geq n$, then $U(\chi^{B^*}, p) > U(\chi^{B^*} - \chi^j, p)$. This implies

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j)$$

Since $u$ is submodular and $\chi^A + \chi^j \in \operatorname{dom} u$, $\chi^A + \chi^j \leq \chi^{B^*}$, we know $u$ has decreasing marginal returns and it follows that

$$p^j < u(\chi^{B^*}) - u(\chi^{B^*} - \chi^j) \leq u(\chi^A + \chi^j) - u(\chi^A)$$

That is, $U(\chi^A, p) < U(\chi^A + \chi^l, p)$. This completes the proof. □

**Proof of Lemma 1:** By the equivalence result, it suffices to show that $\underline{u}(k, \cdot)$ satisfies (SI). Fix a price vector $p$ and $x \notin D_{\underline{u}(k,\cdot)}(p)$. Without loss of generality, we can assume that $p \geq 0$ since, by monotonicity, goods with strictly negative prices will always be demanded. Since the domain of $\underline{u}(k, \cdot)$ is nonempty, there exists $s$ such that $\sigma(s) \geq k$ and $u(s) = \underline{u}(k, \cdot)$ and $s$ is optimal at price $p$ given utility function $\underline{u}(k, \cdot)$.

If $x \notin \operatorname{dom}(\underline{u}(k, \cdot))$, we are done. Now suppose $x \in \operatorname{dom}(\underline{u}(k, \cdot))$, then $\sigma(x) \geq k$. We need to show there exists $y$ such that $\underline{U}(k, x, p) < \underline{U}(k, y, p)$, $\#(\operatorname{supp}(x) - \operatorname{supp}(y)) \leq 1, \#(\operatorname{supp}(y) - \operatorname{supp}(x)) \leq 1$.

If $\sigma(x) = k$, then we have $s$ with $\sigma(s) \geq k$ and $U(s, p) > U(x, p)$. By Lemma B2, we are done.

If $\sigma(x) > k$, then $\sigma(z) \geq k$ for any $z$ such that $\#(\text{supp}\,(x) - \text{supp}\,(z)) \leq 1$ and $\#(\text{supp}\,(z) - \text{supp}\,(x)) \leq 1$. Notice that $u$ satisfies (SI) and $x \notin D_u(p)$, $x \in \text{dom}\,(\underline{u}(k,\cdot)) \subseteq \text{dom}\,u$, then since $u$ satisfies (SI) there exists $y$ such that $U(x,p) < U(y,p)$, $\#(\text{supp}\,(x) - \text{supp}\,(y)) \leq 1, \#(\text{supp}\,(y) - \text{supp}\,(x)) \leq 1$. We know $\sigma(y) \geq k$ and thus $\underline{U}(k,x,p) = U(x,p) < U(y,p) = \underline{U}(k,y,p)$. This completes the proof. □

### 6.2 Proof of Lemma 2

This is a corollary of Theorem 8.2 in Shioura and Tamura (2015). Although they assume $\{0, \chi^H\} \in \text{dom}\,u_i$, their proof can be applied under our assumption that there exists a feasible allocation $\xi$ such that $\sum \xi_i \leq \chi^H$ and $\xi_i \in \text{dom}\,u_i$ for all $i$. By monotonicity, our assumption implies that there exists $\xi^*$ such that $\xi_i^* \in \text{dom}\,u_i$, $\sum_{i=1}^N \xi_i^* = \chi^H$ and

$$\sum_{i=1}^N u_i(\xi_i^*) = \max\big\{\sum_{i=1}^N u_i(x_i) | x_i \in \text{dom}\,u_i, \forall i, \sum_{i=1}^N x_i = \chi^H\big\}$$

Then, the remainder of the proof follows from Theorem 8.2 in Shioura and Tamura (2015).
□

### 6.3 Proof of Theorem 1

Our existence proof relies on Lemma 2, a modification of Kelso and Crawford's proof of existence of an equilibrium for the unconstrained economy with substitutes. In an unconstrained economy, the set of equilibrium prices and the set of equilibrium allocations of divisible goods are independent of the initial endowments. Therefore, we suppress endowments in the definition of the unconstrained economy and write $\mathcal{E}_o = \{u_1, \ldots, u_k\}$ and describe the consumers' problem as maximizing (over $x$)

$$U_i(x,p) = u_i(x) - p \cdot x$$

By assumption, $w_i \in \text{dom}\,u_i$ for each $i$, then Lemma 2 establishes the existence of an equilibrium in deterministic allocations for the unconstrained gross substitutes economy. It is easy to see that for any given an allocation $\alpha$, the set of prices that support $\alpha$; that is, $p$ such that $(p, \alpha)$ is a an equilibrium, is defined by a finite set of linear weak inequalities and therefore is a compact and convex set. Since we are in a transferable utility setting, any Pareto efficient allocation must maximize total surplus. It is also easy to verify that if $(p, \xi)$ is a deterministic equilibrium, and $\hat{\xi}$ is a social surplus maximizing allocation, then $(p, \hat{\xi})$ is also a Walrasian equilibria. The following *exchangeability property* is a consequence of the last three observations: if $(p, \alpha)$ and $(\hat{p}, \hat{\alpha})$ are both equilibria, then $(p, \hat{\alpha})$ is also an equilibrium. Then, it also follows that the set of random equilibrium allocations is simply the convex hull of the set of deterministic equilibrium allocations and hence, the set of equilibrium prices for random allocations is the same as the set of equilibrium prices for deterministic allocations. It follows that for any unconstrained economy, there is a set of prices $P^*$ and a set of random an allocations $\mathcal{A}^*$ such that the set of equilibria is $P^* \times \mathcal{A}^*$.

Since every price in $P^*$ supports the same allocation, $P^*$ is a nonempty, convex and compact set as it is defined by a finite set of linear weak inequalities. Since $\mathcal{A}^*$ is the set of surplus maximizing allocations, it is also a a nonempty, convex and compact set. We summarize these observation in Lemma B3 below.

**Lemma B3:** *For any unconstrained gross substitutes economy $\mathcal{E}_o$, the set of equilibria is $P^* \times \mathcal{A}^*$ for some nonempty compact convex set of prices $P^*$ and some nonempty compact convex set of efficient random allocations $\mathcal{A}^*$.*

Let $\mathcal{A}_o$ be the set of all feasible random allocations for the unconstrained economy $\mathcal{E}_o$ which lie in the product of convex hull of everyone's effective domains. Focusing on $\mathcal{A}_o$ is without loss of generality, as for any $\alpha \notin \mathcal{A}_o$, there must be some agent with utility $-\infty$ and $\alpha$ will be inefficient. Since the set of deterministic feasible allocations is finite and $w_i \in \operatorname{dom} u_i$ for each $i$, $\mathcal{A}_o$ is a nonempty, compact, convex subset of a Euclidian space. For any $\lambda = (\lambda_1, \ldots, \lambda_N) \in [0,1]^N$, we define the maximization problem:

$$M(\lambda) = \max_{\alpha \in \mathcal{A}_o} \sum_i \lambda_i u_i(\alpha_i)$$

Specifically, we assume that $-\infty \times 0 = -\infty$, that is, when $\lambda_i = 0$, $\lambda_i u_i(\cdot)$ has the same effective domain as $u_i$ and is 0 on the effective domain.

Note that $M(\lambda)$ is a linear programming problem. Let $D(\lambda)$ denote the set of solutions to this problem. Hence, the set $D(\lambda)$ is nonempty, compact and convex. By Berge's Theorem, the correspondence $D$ is upper-hemi continuous (uhc). That is, $D$ is a nonempty, compact and convex-valued, uhc correspondence.

For $\lambda \in [0,1]^N$, define the unconstrained economy $\mathcal{E}_o(\lambda) = \{\lambda_1 u_1, \ldots, \lambda_N u_N\}$. Note that the unconstrained demand of $u^j$ at price $p$ is same as the unconstrained demand of $\lambda_i u_i$ at price $\lambda_i p_i$ and hence $\mathcal{E}_o(\lambda)$ is an unconstrained gross substitutes economy.

By Lemma B3, $D(\lambda)$ is the set of equilibrium allocations for the economy $\mathcal{E}_o(\lambda)$. Let $a = \sum_i u_i(\chi^H)$ and $\mathcal{P} = [0,a]^N$. Hence, any equilibrium price $p$ of the unconstrained economy $\mathcal{E}_o(\lambda)$ must be in $\mathcal{P}$. Let $P^*(\lambda)$ be the set of all equilibrium prices for $\mathcal{E}_o(\lambda)$. Then, by Lemma B3 above, $\emptyset \neq P^*(\lambda) \subset \mathcal{P}$. Also by Lemma B3, the set of equilibrium prices is nonempty, convex and compact.

**Lemma B4:** *For any constrained economy $\mathcal{E}$, there exists a $\lambda \in [0,1]^N$ and an equilibrium $(p,\alpha)$ of a corresponding unconstrained economy $\mathcal{E}_o(\lambda)$ such that if $\lambda_i < 1$, then $p \cdot \bar{\alpha}_i = b_i$ and if $\lambda_i = 1$, then $p \cdot \bar{\alpha}_i \leq b_i$.*

**Proof:** Let $U_i^\lambda$ be consumer $i$'s utility function in the unconstrained economy $\mathcal{E}_o(\lambda)$; that is, $U_i^\lambda(\theta, p) = \lambda_i u_i(\theta) - p \cdot \bar{\theta}$.

It is easy to verify that the correspondence, $D(\cdot)$ is upper hemi-continuous. By Lemma B3, it is also convex valued. Again, by Lemma B3, $P^*(\cdot)$ is also convex valued. Next, we will show that it is upper hemi-continuous as well. Since $P^*(\cdot)$ is compact-valued (by Lemma B3), it is enough to show that $\lambda(t) \in [0,1]^N$, $p(t) \in P^*(\lambda(t))$ for all $t = 1, 2, \ldots$, $\lim \lambda(t) = \lambda$ and $\lim p(t) = p$ implies $p \in P^*(\lambda)$.

Choose $\alpha(t) \in D(\lambda(t))$ for all $t$. Since $\mathcal{A}$ is compact, we can assume, by passing to subsequence if necessary, that $\alpha(t)$ converges. Let $\alpha = \lim \alpha(t)$. Since $D(\cdot)$ is upper hemi-continuous, $\alpha \in D(\lambda)$.

Let $\beta$ be any feasible allocation. Then, the efficiency of $\alpha$ and the feasibility of $\beta$ imply

$$U_i(\alpha_i(t), p(t)) = u_i(\alpha_i(t)) - p(t) \cdot \bar{\alpha}_i(t) \geq u_i(\beta_i) - p(t) \cdot \bar{\beta}_i = U_i(\beta_i, p(t))$$

Then, the continuity of $U_i$ ensures that $U_i(\alpha_i, p) \geq U_i(\beta_i, p)$ for all $\beta$ and for all $i$. This implies that $p \in P^*(\lambda)$ and establishes the upper hemi-continuity of $P^*$.

Next, define correspondence $\Gamma_i$ as follows:

$$\Gamma_i(p,z) = \begin{cases} [0,1] & \text{if } p \cdot (z - w_i) = b_i \\ 0 & \text{if } p \cdot (z - w_i) > b_i \\ 1 & \text{if } p \cdot (z - w_i) < b_i \end{cases}$$

Clearly, $\Gamma_i$ is nonempty, convex and compact valued, and upper hemi-continuous.

Let $\Omega = \mathcal{P} \times \mathcal{A}_o \times [0,1]^N$ and let

$$f(p, \alpha, \lambda) = P^*(\lambda) \times D(\lambda) \times \Gamma_1(p, \bar{\alpha}_1) \times \cdots \times \Gamma_N(p, \bar{\alpha}_N)$$

Since $P^*$, $D$ and the $\Gamma_i$'s are nonempty, convex and compact valued, and upper hemicontinuous and the mapping $\alpha \to \bar{\alpha}_i$ is continuous, $f$ is also nonempty, convex and compact valued, and upper hemicontinuous. Then, by Kakutani's Fixed-Point Theorem, there is an $\omega^* = (p^*, \alpha^*, \lambda^*)$ such that $f(\omega^*) = \omega^*$. Thus, $(p^*, \alpha^*)$ is a Walrasion equilibrium of economy $\mathcal{E}_o(\lambda^*)$.

We claim that $\lambda_i^* > 0$ for all $i$. To see why, note that if $\lambda_i^* = 0$, then since $\omega^*$ is a fixed point of $f$, $p \cdot (\bar{\alpha}_i^* - w_i) \geq b_i > 0$. Since $\lambda_i^* = 0$, agent $i$'s utility is identically 0 on the effective domain of $u_i$. But since $w_i \in \operatorname{dom} u_i$, demanding $w_i$ would yield strictly greater utility than $\alpha_i^*$, a contradiction.

To complete the proof of the lemma, we will show that $p^* \cdot (\bar{\alpha}_i^* - w_i) \leq b_i$ for all $i$ and that the inequality is an equality whenever $\lambda_i^* < 1$. Since $\omega^*$ is a fixed point of $f$ and $\lambda_i^* > 0$, we must have $p^* \cdot (\bar{\alpha}_i^* - w_i) \leq b_i$. Similarly, since $\omega^*$ is a fixed point of $f$, if $\lambda_i^* < 1$, we must have $p^* \cdot (\bar{\alpha}_i^* - w_i) = b_i$. □

To conclude the proof, we will show that $(p^*, \alpha^*)$ is an equilibrium of the constrained economy $\mathcal{E}$. Suppose for some $i$ there is an affordable $\theta$ (in the constrained economy) such that

$$U_i(\theta, p^*) = u_i(\theta) - p^* \cdot \bar{\theta} > u_i(\alpha_i^*) - p^* \cdot \bar{\alpha}_i^* \tag{2}$$

Since $\alpha^*$ is an equilibrium of the unconstrained economy, $\mathcal{E}_o(\lambda^*)$, we must have $\lambda_i^* < 1$. (Otherwise, equation (2) would contradict the optimality of $\alpha_i^*$ for $i$ in the unconstrained economy.) The optimality of $\alpha_i^*$ for $i$ in the unconstrained economy implies

$$\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^* \tag{3}$$

Since $\lambda_i^* < 1$ and $\omega^*$ is a fixed-point of $f$, the right-hand side of equation (3) must be less than or equal to zero. Then, we have $\lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq 0$ and hence $u_i(\theta) - u_i(\alpha_i^*) \leq \lambda_i^*(u_i(\theta) - u_i(\alpha_i^*)) \leq p^* \cdot \bar{\theta} - p^* \cdot \bar{\alpha}_i^*$ contradicting equation (2). □

### 6.4 Proof of Theorem 2

Fix the no transfer payments economy $\mathcal{E}^* = \{(u_i, b_i)_{i=1}^N\}$ and define the sequence of constrained economies $\mathcal{E}_n = \{(nu_i, w_i, b_i)_{i=1}^N\}$ for $n = 1, 2\ldots$ where $w_i^j = 0$ for all $j, i$. Notice that $0 \in \operatorname{dom} u_i$ for all $i$, by Theorem 1, each $\mathcal{E}_n$ has an equilibrium $(p^n, \alpha^n)$[17]. By monotonicity, $0 \in \operatorname{dom} u_i$ implies that $\operatorname{dom} u_i = X$ and we need not worry about the issue of effective domains. Let $P = [0, \sum_i b_i]^L$. Note that $p^n$ must be an element of $P$. Hence, the sequence $(p^n, \alpha^n)$ lies in a compact set and therefore has a limit point, $(p, \alpha)$. By passing to a subsequence if necessary, we assume that $(p, \alpha)$ is its limit. To conclude the proof, we show that this limit point must be a strong equilibrium of $\mathcal{E}^*$.

Clearly, $\alpha$ is feasible for $\mathcal{E}^*$ and $p^n \cdot \bar{\alpha}_i^n \leq b_i$ for all $n$ implies $p \cdot \bar{\alpha}_i \leq b_i$ for all $i$. Hence, $\alpha_i$ is affordable for $i$ in $\mathcal{E}^*$. Take any other affordable random allocation $\theta$ for $i$ in $\mathcal{E}^*$, that is, $p \cdot \bar{\theta} \leq b_i$, we need to show that $u_i(\theta) \leq u_i(\alpha_i)$. First, consider the case where $p \cdot \bar{\theta} < b_i$. Such $\theta$ exists since $b_i > 0$ and $p \geq 0$. Then there exists $\epsilon > 0$ such that for any $p' \in B(p, \epsilon) \cap I\!R_+^L$, where $B(p, \epsilon)$ is the $\epsilon$-ball centered at $p$, $p' \cdot \bar{\theta} < b_i$. Since $p^n \to p$, we can find $M > 0$ such that $\forall n \geq M$, $p^n \cdot \bar{\theta} < b_i$. This implies $\theta$ is affordable for $i$ in $\mathcal{E}_n$ for $n \geq M$. By optimality of $\alpha_i^n$, $nu_i(\theta) - p^n \cdot \bar{\theta} \leq nu_i(\alpha_i^n) - p \cdot \bar{\alpha}_i^n$. Hence, $u_i(\theta) - u_i(\alpha_i^n) \leq (p^n \cdot \bar{\theta} - p^n \cdot \bar{\alpha}_i^n)/n \leq b_i/n$ for all $n \geq M$. By continuity, let $n \to \infty$, we know $u_i(\theta) \leq u_i(\alpha_i)$ for all $p \cdot \bar{\theta} < b_i$. Now we consider the case with $p \cdot \bar{\theta} = b_i$. Then there exists a sequence of $\theta^n$ such that $p \cdot \bar{\theta}^n < b_i$ and $\theta^n \to \theta$. By the previous case and continuity, again we know $u_i(\theta) \leq u_i(\alpha_i)$. Thus, $\alpha_i$ is optimal for agent $i$ under $p$ in $\mathcal{E}^*$. Last, to show all goods with strictly positive prices are allocated to agents, it is equivalent to show $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$. This is true since $p^{n,j} \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^{n,j}) = 0$ for all $j, n$ and continuity holds. Thus, $(p, \alpha)$ is an equilibrium of $\mathcal{E}^*$.

To conclude, we will show that $(p, \alpha)$ is a strong equilibrium; that is, for all $i$, $u_i(\theta) = u_i(\alpha_i)$ implies $p \cdot \bar{\theta} \geq p \cdot \bar{\alpha}_i$. If not, assume that $p \cdot \bar{\theta} < p \cdot \bar{\alpha}_i$ for some $\theta$ such that $u_i(\theta) = u_i(\alpha_i)$ and consider two cases: (1) agent $i$ is satiated at $\theta$; that is, $u_i(\theta) = u_i(\alpha_i) = u_i(\chi^H)$ or (2) she is not satiated at $\theta$.

[17] Without confusion, we use superscripts to represent both identity of goods (with generic element $j$) and identity of an element in a sequence of prices or allocations (with generic element $n, m$). Sometimes double superscripts are used

If (1) holds, then for sufficiently large $n$, purchasing $\theta$ instead of $\alpha_i$ is affordable for $i$ at all $p^n$ and $b_i - p\cdot\bar{\theta} > b_i - p\cdot\bar{\alpha}_i^n \geq 0$, contradicting the optimality of $\alpha_i^n$ for $i$ in $\mathcal{E}_n$. If (2) holds, then choose $0 < r < 1$ such that $p\cdot(r\chi^H + (1-r)\bar{\theta}) < p\cdot\bar{\alpha}_i$. Again, for $n$ sufficiently large, the random consumption $r\delta_{\chi^H} + (1-r)\theta$, where $\delta_{\chi^H}$ is the degenerate lottery that yields $\chi^H$ for sure, is affordable at $p^n$ and yields a higher utility than $\alpha_i^n$, contradicting its optimality in $\mathcal{E}_n$. $\square$

### 6.5 Proof of Lemma 3 and Theorem 3

**Proof of Lemma 3:** By assumption, the effective domain of $u(c,\cdot)$ is nonempty. We will prove that $u(c,\cdot)$ satisfies the substitutes property whenever $u$ satisfies the substitutes property and $c = \{(A(k),(l(k),h(k))_{k=1}^K\}$ is a modular constraint set for $u$ by induction on $K$. Suppose $K = 1$ and hence, $c = \{(A,l,h)\}$. Recall that both the $\chi^A$-constrained and the $\chi^{A^c}$-constrained $u$ satisfy the substitutes property since $u$ satisfies it. Therefore, $v_0$, the $h$-satiation of the $\chi^A$-constrained $u$, and $v$, the $l$-lower-bound of $v_0$, are both substitutes utility functions. Since $A$ is a module, $u(c,\cdot)$ is the convolution of $v$ and the $\chi^{A^c}$-constrained $u$ it satisfies the substitutes property.

Suppose the result holds for $K-1$ and consider $c = \{(A(k),(l(k),h(k))_{k=1}^K\}$.

*Case 1.* Suppose that there does not exist a set in $\{A(k)\}_{k=1}^K$ such that $A(k) \subseteq A(k^*)$ for all $k$. Let $A(k)$ be a maximal set in $\{A(k)\}_{k=1}^K$ (that is, there is no $i \neq k$ such that $A(k) \subset A(i)$) and there exists $k' \neq k$ with $A(k)\cap A(k') = \emptyset$.) Define $I_1 = \{i : A(i) \subseteq A(k)\}$, $I_2 = \{i : A(i)\cap A(k) = \emptyset\}$. Since $A(k)$ is maximal and $\{A(k)\}_{k=1}^K$ form a hierarchy, $I_1, I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1,...,K\}$. Notice that $A(k)$ is a module of $u$, $u(x) = u(x\wedge\chi^{A(k)}) + u(x\wedge\chi^{A(k)^c})$ and both $\chi^{A(k)}$-constrained and the $\chi^{A(k)^c}$-constrained $u$ satisfy the substitutes property. As $|I_1| < K$, $|I_2| < k$ and $\{A_i\}_{i\in I_1}, \{A_i\}_{i\in I_2}$ are both hierarchies, by the inductive hypothesis, $\chi^{A(k)}$-constrained $u$ with constraints $c_1 = \{(A(i),l(i),h(i))_{i\in I_1}\}$ (denoted by $v_1$) and $\chi^{A(k)^c}$-constrained $u$ with constraints $c_2 = \{(A(i),l(i),h(i))_{k\in I_2}\}$ (denoted by $v_2$) both satisfy substitutes. Moreover, since constraints in $I_1$ only matter for goods within $A(k)$ and constraints in $I_2$ only matter for goods within $A(k)^c$, $u(c,\cdot)$ is the convolution of $v_1$ and $v_2$ and thus satisfies substitutes property.

*Case 2.* Suppose, without loss of generality, that $A(i) \subseteq A(K)$ for all $i = 1, ..., K-1$. Let $B = \cup_{i=1}^{K-1} A(i) \subseteq A(K)$. First, note that the $\chi^B$-constrained $u$, the $\chi^{A(K)-B}$-constrained $u$, and the $\chi^{A(K)^c}$-constrained $u$ satisfy the substitutes property. By the inductive hypothesis $u(c', \cdot)$ with $c' = \{(A(i), l(i), h(i))_{i=1}^{K-1}\}$ satisfies the substitutes property. Thus, $v_3$ defined as the $\chi^B$-constrained $u(c', \cdot)$ satisfies the substitutes property. Let $v_4$ be the convolution of $v_3$ and the $\chi^{A(K)-B}$-constrained $u$ and let $v_5$ be $v_4$ with constraint $(A(K), l(K), h(K))$. Then, $v_4$ and $v_5$ satisfy the substitutes property. Finally, $u(c, \cdot)$ is the convolution of $v_5$ and the $\chi^{A(K)^c}$-constrained $u$ and, thus, satisfies substitutes property. □

**Proof of Theorem 3:** Fix the no transfer payments economy with modular constraint $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$. By definition, there exists $x_1, \cdots, x_{N+1}$ such that $x_i \in \operatorname{dom} u_j(c_j, \cdot)$ for all $i = 1, ..., N+1$, $j = 1, ..., N$. Note that $u_j(c_j, \cdot)$ is monotone since $u_j$ is monotone, and, therefore, we can assume $\sum x_i = \chi^H$. Let $\alpha^0$ be a random allocation such that $\alpha_i^0 = \theta^0$ for all $i = 1, \ldots, N$ and $\theta^0(x_j) = 1/N$ for all $j = 1, \ldots, N-1$ and $\theta^0(x_N + x_{N+1}) = 1/N$. By monotonicity again, every realization of $\theta^0$ lies in the effective domain of every agent. This implies $\bar{\theta}^{0,j} = 1/N$ for all $j \in H$. Moreover, we can define another feasible random consumption for all agents as $\theta^1(x_j) = 1/(N+1)$ for all $j = 1, \ldots, N+1$ and $\bar{\theta}^{1,j} = 1/(N+1)$ for all $j \in H$.

Consider the following constrained economy with random endowments. Each consumer $i$ has the utility $u_i(c_i, \cdot)$, the random endowment $\theta^0$ and $b_i = 1$. The budget set of consumer $i$ is

$$B(p, \theta^0, 1) = \{\theta \in \Theta | \, p(\bar{\theta} - \bar{\theta}^0) \leq 1\}$$

Otherwise, the definition of the economy is identical to the definition of a constrained economy. It is straightforward to extend existence of equilibrium in Theorem 1 to this economy since every realization of the (random) endowment is in the effective domain of $u_i(c_i, \cdot)$ for all $i$.

Next, define the sequence of constrained economies $\mathcal{E}_n = \{(nu_i(c_i, \cdot), \theta^0, 1)_{i=1}^N\}$ for $n = 1, 2 \ldots$. By Lemma 3, for each $i$, $nu_i(c_i, \cdot)$ satisfies substitutes property. Then, each $\mathcal{E}_n$ has an equilibrium $(p^n, \alpha^n)$. By monotonicity, $p^n \geq 0$ and we can assume that $\sum_{i=1}^N \bar{\alpha}_i^n = \chi^H$. We distinguish two cases:

**Case 1:** There is a subsequence such that $p^n$ is bounded.

Let $(p^*, \alpha)$ be a limit point of $(p^n, \alpha^n)$ and let $p = \lambda p^*$ such that $\lambda = 1/(1 + p^*\theta^0)$. We will show that this $(p, \alpha)$ must be an equilibrium of the no transfer payments economy with modular constraint $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$. Clearly, $\alpha$ is feasible for $\mathcal{E}_c^*$ and $p^n \cdot (\bar{\alpha}_i^n - \bar{\theta}^0) \leq 1$ for all $n$ implies $p \cdot \bar{\alpha}_i \leq 1$ for all $i$ by the definition of $\lambda$. Hence, $\alpha_i$ is affordable for $i$ in $\mathcal{E}^*$.

Take any other affordable random allocation $\theta$ for $i$ in $\mathcal{E}^*$, that is, $p \cdot \bar{\theta} \leq 1$, we need to show that $u_i(c_i, \theta) \leq u_i(c_i, \alpha_i)$. First, consider the case where $p \cdot \bar{\theta} < 1$. Then there exists $\epsilon > 0$ such that for any $p' \in B(p, \epsilon) \cap I\!R_+^L$, where $B(p, \epsilon)$ is the $\epsilon$-ball centered at $p$, $p' \cdot \bar{\theta} < 1$. Since $p^n \to p^*$, we can find $M > 0$ such that $\forall n \geq M$, $p^n \cdot (\bar{\theta} - \bar{\theta}^0) < 1$. This implies $\theta$ is affordable for $i$ in $\mathcal{E}_n$ for $n \geq M$. By optimality of $\alpha_i^n$, $nu_i(c_i, \theta) - p^n \cdot \bar{\theta} \leq nu_i(c_i, \alpha_i^n) - p \cdot \bar{\alpha}_i^n$. Hence, $u_i(c_i, \theta) - u_i(c_i, \alpha_i^n) \leq (p^n \cdot \bar{\theta} - p^n \cdot \bar{\alpha}_i^n)/n$ for all $n \geq M$. Since $\{p^n\}$ is bounded, the right hand side goes to 0. Passing to the limit, continuity then implies $u_i(c_i, \theta) \leq u_i(c_i, \alpha_i)$ for all $p \cdot \bar{\theta} < 1$.

Second, consider the case with $p \cdot \bar{\theta} = 1$. If $u(\theta) = -\infty$, then we are done. Now suppose $u(\theta) \in I\!R$, that is, $\theta$ is the randomization of allocations in the effective domain of $u$. Notice that $u(\theta^0) \in I\!R$ and $p \cdot \bar{\theta}^0 < 1$. Then there exists a sequence of $\theta^n = (1/n)\theta^0 + (1 - 1/n)\theta$ such that $p \cdot \bar{\theta}^n < 1$, $u(\theta_n) \in I\!R$ and $\theta^n \to \theta$. By the previous case and continuity, again we know $u_i(c_i, \theta) \leq u_i(c_i, \alpha_i)$. Thus, $\alpha_i$ is optimal for agent $i$ under $p$ in $\mathcal{E}^*$.

Finally, to show all goods with strictly positive prices are allocated to agents, it is sufficient to show that $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$. This is true since $p^{n,j} \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^{n,j}) = 0$ for all $j, n$ and, therefore, $p^j \cdot (1 - \sum_{i=1}^N \bar{\alpha}_i^j) = 0$, by continuity of the left hand side expression. Thus, $(p, \alpha)$ is an equilibrium of $\mathcal{E}^*$.

**Case 2:** $\sum_{j=1}^L p^{n,j} \to \infty$.

Let $\hat{p}^n = N \cdot p^n / (\sum_{k=1}^L p^{n,k})$ and note that $\hat{p}^n \in [0, N]^L$. Let $(p, \alpha)$ be the limit of a convergent subsequence of $(\hat{p}^n, \alpha^n)$. When there is no confusion, we say $(\hat{p}^n, \alpha^n)$ converges to $(p, \alpha)$. We will show that this $(p, \alpha)$ must be an equilibrium of the no transfer payments economy with modular constraint $\mathcal{E}_c^* = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N\}$ . Clearly, $\alpha$ is feasible for $\mathcal{E}^*$. To see that it is affordable, note that

$$p^n \cdot \bar{\alpha}_i^n / p^n \bar{\theta}^0 \leq (1 + p^n \bar{\theta}^0) / p^n \bar{\theta}^0$$

for all $n$. Since $\bar{\theta}^0(k) = 1/N$ for all $k$ and,

$$(1 + p^n\bar{\theta}^0)/p^n\bar{\theta}^0 \to 1$$

it follows that $p \cdot \bar{\alpha}_i \leq 1$ and thus $\alpha_i$ is affordable at $p$ as desired.

By construction, we have that for each $n$,

$$\sum_{i=1}^{N} \hat{p}^n \cdot \bar{\alpha}_i^n = \hat{p}^n \cdot \chi^H = N$$

Passing to the limit then implies that

$$\sum_{i=1}^{N} p \cdot \bar{\alpha}_i = N$$

Since $p \cdot \bar{\alpha}_i \leq 1$ for each $i$, this implies $p \cdot \bar{\alpha}_i = 1$ for all $i$.

Consider any affordable random allocation $\theta$, that is, $p \cdot \bar{\theta} \leq 1$, we must show that $u_i(c_i, \theta) \leq u_i(c_i, \alpha_i)$. First, consider the case where $p \cdot \bar{\theta} < 1 = p \cdot \bar{\alpha}_i$, then there exists $\epsilon > 0$ such that for any $p' \in B(p, \epsilon) \cap I\!R^L_+$ and $\alpha' \in B(\alpha_i, \epsilon) \cap \Theta$, (recall that $\Theta$ is the set of all random consumptions), $p' \cdot \bar{\theta} < p' \cdot \bar{\alpha'}$. Since $\hat{p}^n \to p$ and $\alpha_i^n \to \alpha_i$, we can find $M > 0$ such that $\forall n \geq M$, $\hat{p}^n \cdot \bar{\theta} < \hat{p}^n \cdot \bar{\alpha}_i^n$, that is, $p^n \cdot \bar{\theta} < p^n \cdot \bar{\alpha}_i^n$. Then clearly $\theta$ is affordable for $i$ in $\mathcal{E}_n$ for $n \geq M$ as $\alpha_i^n$ is affordable. By optimality of $\alpha_i^n$, $nu_i(c_i, \theta) - p^n \cdot \bar{\theta} \leq nu_i(c_i, \alpha_i^n) - p \cdot \bar{\alpha}_i^n$. Hence, $u_i(c_i, \theta) - u_i(c_i, \alpha_i^n) \leq (p^n \cdot \bar{\theta} - p^n \cdot \bar{\alpha}_i^n)/n < 0$ for all $n \geq M$. Passing to the limit, continuity then implies $u_i(c_i, \theta) \leq u_i(c_i, \alpha_i)$ for all $p \cdot \bar{\theta} < 1$. For the case with $p \cdot \bar{\theta} = 1$, recall that $u_i(\theta^1) \in I\!R$ for all agents $i$ and $p \cdot \bar{\theta}^1 < p \cdot \bar{\theta}^0 = 1$, then the result follows from the same argument as in Case 1.

Finally, we will show that (in either case) $(p, \alpha)$ is a strong equilibrium given the constraints; that is, for all $i$, $u_i(c_i, \theta) = u_i(c_i, \alpha_i)$ implies $p \cdot \bar{\theta} \geq p \cdot \bar{\alpha}_i$. If not, assume that $p \cdot \bar{\theta} < p \cdot \bar{\alpha}_i$ for some $\theta$ such that $u_i(c_i, \theta) = u_i(c_i, \alpha_i)$. Then for sufficiently large $n$, purchasing $\theta$ instead of $\alpha_i$ is affordable for $i$ at all $p^n$ and $-p \cdot \bar{\theta} > -p \cdot \bar{\alpha}_i^n$, contradicting the optimality of $\alpha_i^n$ for $i$ in $\mathcal{E}_n$. □

### 6.6 Proof of Theorem 4 and Lemma 4

**Proof of Lemma 4**

We order $d$, the hierarchy of constraints, in the obvious way: $(A,k) \succ (B,n)$ if $A \neq B$ and $B \subset A$. To prove the lemma, we will show that $\mathcal{I}_d$ is a matroid. This will imply that the set of elements of $\mathcal{I}_d$ with the maximal cardinality is a basis system. Clearly, $0 \in \mathcal{I}_d$ and $x,y \in \mathcal{I}_d$ and $x \leq y$ implies $x \in \mathcal{I}_d$. Hence, we need only prove that $x,y \in \mathcal{I}_d$ and $\sigma(x) < \sigma(y)$ implies there is $j$ such that $x^j < y^j$ and $x^j + \chi^j \in \mathcal{I}_d$.

Call $j$ a *free* element in $d$ if $j$ is not an element of any $A$ such that $(A,n) \in d$ for some $n$. Otherwise, call $j$ a constraint element. Let $F \subset H$ be the set of free elements in $d$ and let $F^c = H \backslash F$ be the set of constraint elements. Suppose there is $j \in F$ such that $y^j > x^j$. Then, clearly $x + \chi^j \in \mathcal{I}_d$ and we are done. Otherwise, $x^j \geq y^j$ for all $j \in F$ and hence there must be some maximal constraint, $(A,n)$, (with respect to the binary relation $\succ$ define above) such that $\sigma(y \wedge \chi^A) > \sigma(x \wedge \chi^A)$. Let $A_1 = A$.

Then, there is either $j \in A_1$ such that $y^j > x^j$, $j \notin B$ for any $B$ such that $(A_1, n_1) \succ (B, n')$ in which case we have $x + \chi^j \in \mathcal{I}_d$ and we are done, or there is no such $j$. In the latter case, there must be a maximal element of the set $\{B \,|\, A_1 \succ B\}$ such that $\sigma(y \wedge \chi^B) > \sigma(x \wedge \chi^B)$. Let $A_2 = B$. Continuing in this fashion, by finiteness, we are either done or end up with a maximal chain $(A_1, n_1) \succ \cdots \succ (A_l, n_l)$ such that $\sigma(y \wedge \chi^{A_k}) > \sigma(x \wedge \chi^{A_k})$ for all $k = 1, \ldots, l$. Choose any $j \in A_l$ such that $y^j > x^j$ and note that $x + \chi^j \in \mathcal{I}_d$. □

**Proof of Theorem 4**

Fix an economy substitutes economy with matroidal technology $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$. Let

$$H = \{1, \ldots, L\} = \{i \,|\, x_i > 0 \text{ for some } x \in \mathcal{B}\}$$

Let $\mathcal{B}^\perp = \{\chi^H - x \,|\, x \in \mathcal{B}\}$. Then $\mathcal{B}^\perp$ is a basis system whenever $\mathcal{B}$ is a basis system (Appendix A); Let $r$ be the rank function associated with $\mathcal{B}^\perp$; that is,

$$r(x) = \max\{\sigma(x \wedge y) \,|\, y \in \mathcal{B}^\perp\}$$

Since every weighted matroid satisfies the substitutes property, so does $r$ (Appendix A). Now, consider the sequence of $N+1$ person no transfer payments economies without production $\mathcal{E}_n = \{(u_i, b_i)_{i=1}^{N+1}\}$ with aggregate endowment $\{1, \ldots, L\}$, $u_{N+1} = r$ and $b_{N+1} = n$.

Let $(p^n, \alpha^n)$ be a strong equilibrium for economy $\mathcal{E}_n$. Without loss of generality, we assume that $\alpha^n$ converges and each $p^{nj}$ converges either to a real number or to infinity.

Let $n^*$ be the cardinality of the support of vectors in $\mathcal{B}^\perp$ and let $\hat{n}$ be the cardinality of $\mathcal{B}^\perp$. Clearly, $n^*$ is well defined as $\mathcal{B}^\perp$ is a basis system and $\hat{n} > 2$ as $|\mathcal{B}| > 2$. We claim that there exists $m$ such that for all $n > m$, the equilibrium utility of agent $N+1$ is maximal; that is, equal to $n^*$. To see why this is the case, first assume that $\lim p^{n,j} < \infty$ for all $j$. Then, it is immediate that $N+1$ utility is maximal since she can afford to buy the entire endowment in $\hat{e}_n$ whenever $n > \sum_j \lim p^{n,j}$.

Next, assume that $\lim p^{n,j} = \infty$ for all $j$. Then, agent $N+1$ must be consuming each $j$ with arbitrarily high probability as $n$ approaches infinity. In particular, there must be some $m$ such that for all $n > m$ the probability with which agent $N+1$ consumes $j$; that is, $\bar{\alpha}_i^j$, must be greater than $(\hat{n}-1)/\hat{n}$. This means that agent $N+1$ can afford the random consumption $\theta$ such that $\theta(x) = 1/\hat{n}$ for all $x \in \mathcal{B}^\perp$ as for each $j = 1, \ldots, L$, there exists some $y \in \mathcal{B}^\perp$ such that $y^j = 0$, . Clearly, this $\theta$ gives agent $N+1$ her maximal utility.

Finally, there exists at least one $j$ such that $\lim p^{n,j} < \infty$ and some $j$'s such that $\lim p^{n,j} = \infty$, then choose $\epsilon \in (0, 1 - (\hat{n}-1)/\hat{n})$ and arguing as in the second paragraph above, note that the probability with which agent $N+1$ consumes any good in the second category in $\alpha_n$ must be greater than $\epsilon + (\hat{n}-1)/\hat{n}$ for $n$ sufficiently large. By consuming $\epsilon/2$ less of one such good, this agent he will have enough money to consume all of the goods in the first category; that is, the goods such that $\lim p^{n,j} < \infty$, for $n$ sufficiently large. Hence, again, agent $N+1$ can afford the random consumption $\theta$ described above and hence her equilibrium utility must be $n^*$.

Let $\theta^n$ be the equilibrium consumption of agent $N+1$ in economy $\mathcal{E}_n$ for some $n$ large enough to ensure that her payoff is maximal; that is, $u_{N+1}(\theta^n) = n^*$. Since $(p^n, \alpha^n)$ is a strong equilibrium, we can assume, without loss of generality, that agent $N+1$ never demand positive shares for a bundle outside $\mathcal{B}^\perp$. This means that $\theta^n(x) > 0$ implies $x \in \mathcal{B}^\perp$ which means that $\alpha^n(\xi) > 0$ implies $\sum_{i=1}^N \xi_i \leq z$ for some $z \in \mathcal{B}$. Without loss of generality, assume that $y := \sum_{i=1}^N \xi_i = z$. This is without loss of generality since if $y^j < z^j$, then the fact that $\alpha^n$ is a strong equilibrium ensures that $p^n(j) = 0$ in which case we can give good $j$ to any agent $i = 1, \ldots, N$ by monotonicity of utility functions.

Then, to construct a strong equilibrium $(p^n, \beta)$ of $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ and let

$$\beta(x_1, \ldots, x_N, \chi^H - z) = \alpha^n(x_1, \ldots, x_N, z)$$

for every deterministic allocation $(x_1, \ldots, x_n)$ and $z \in \mathcal{B}^\perp$. Since $\alpha^n$ is a strong equilibrium for the $(N+1)$-agent exchange economy $\mathcal{E}_n$, $\alpha_{N+1}^n(z) > 0$ implies $z \in \mathcal{B}^\perp$ is the cheapest optimal consumption for agent $N+1$; that is, $p^n \cdot z \leq p^n \cdot y$ for all $y \in \mathcal{B}^\perp$, no matter whether $y$ is affordable for agent $N+1$ in $\mathcal{E}_n$. This means that $\chi^H - z$ maximizes profit among all $y \in \mathcal{B}$. It is easy to check other requirements of a strong equilibrium hold as well. Hence, $(p^n, \beta)$ is a strong equilibrium of the economy $\tilde{\mathcal{E}} = \{(u_i, b_i)_{i=1}^N, \mathcal{B}\}$ with the matroidal technology $\mathcal{B}$.

The Pareto-efficiency of strong equilibria follows from standard arguments. □

### Proof of Corollary

Given a production economy with no transfers payments and modular constraints $\tilde{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, \{c_i\}_{i=1}^N, \mathcal{B}\}$, we can define $H, \mathcal{B}^\perp, \hat{n}, n^*$ and $r(\cdot)$ as in the proof of Theorem 4. Consider the sequence of $N+1$-person no transfer payments economies with modular constraints and without production $\tilde{\mathcal{E}}_c^n = \{(u_i, 1)_{i=1}^N, (u_{N+1}, b_{N+1}), \{c_i\}_{i=1}^N\}$ with aggregate endowment $\chi^H$, $u_{N+1} = r$ and $b_{N+1} = n$. Agent $N+1$ has no individual constraints and $\operatorname{dom} u_{N+1} = X$. If, for $n$ large enough, there exists a strong equilibrium for economy $\tilde{\mathcal{E}}_c^n$, then by the same argument as in the proof for Theorem 4, we are done. Thus, it suffices to prove the following claim:

**Claim**: *There exists $N^*$ such that for any $n \geq N^*$, there exists a strong equilibrium for economy $\tilde{\mathcal{E}}_c^n$.*

*Step 1:* Denote economy $\hat{\mathcal{E}}_c = \{(u_i, 1)_{i=1}^N, u_{N+1}, \{c_i\}_{i=1}^N\}$ where the only difference from $\tilde{\mathcal{E}}_c^n$ is that agent $N+1$ is not budget constrained. Suppose that $(p, \alpha)$ is a strong equilibrium of $\hat{\mathcal{E}}_c$, and denote $N^* = p \cdot \bar{\alpha}_{N+1}$, then $u_{N+1}(\alpha_{N+1}) = n^*$, that is, agent $N+1$ achieves the maximum possible utility $(p, \alpha)$. This implies that for any $n \geq N^*$, $\alpha_{N+1}$ is affordable for agent $N+1$ in $\tilde{\mathcal{E}}_c^n$ and thus $(p, \alpha)$ is a strong equilibrium in $\tilde{\mathcal{E}}_c^n$. Thus, it suffices to show $\hat{\mathcal{E}}_c$ admits a strong equilibrium.

*Step 2:* Recall that in a constrained economy, all agents have marginal utility 1 for money, while in a no transfer payments economy, all agents have marginal utility 0 for money. Now we want to generalize Theorem 2 to allow for both kinds of agents. Formally, for a value function $u_i$, we denote $u_i^1$ if the marginal utility for money is 1 for agent $i$ and $u_i^0$ if the marginal utility for money is 0. Consider the economy $\bar{\mathcal{E}} = \{(u_i^1, w_i, b_i)_{i=1}^N, (u_{N+1}^0, b_{N+1})\}$ where $w_i \in \operatorname{dom} u_i$ for all $i = 1, ..., N$, $0 \in \operatorname{dom} u_{N+1}$ and $u_i$ satisfies substitutes property for all $i = 1, ..., N+1$. The definition of a strong equilibrium is as before except that the requirement that agents choose the cheapest optimal consumption is only imposed on agents $i = 1, \ldots, N$, i.e., those agents who have 0 marginal utility of money.

To show that a strong equilibrium exists for $\bar{\mathcal{E}}$, we consider the sequence of economies where only agent $N+1$'s utility is multiplied by $n$: $\bar{\mathcal{E}}_n = \{(u_i^1, w_i, b_i)_{i=1}^N, (nu_{N+1}^1, b_{N+1})\}$. By Theorem 1, there exists a equilibrium for each $n$. Since $0 \in \operatorname{dom} u_{N+1}$, the rest of the proof is exactly the same as Theorem 2.

*Step 3:* Construct the sequence of economy as in Theorem 3. Define $w_{N+1}$ as $w_{N+1}(x) = 1/\hat{n}$ for all $x \in \mathcal{B}^\perp$. By part (4) of the definition and monotonicity, for any $x^k \in \mathcal{B}^\perp$, $k = 1, ..., \hat{n}$, there exists $\xi_1^k, ..., \xi_{N+1}^k$ such that $\sum_{i=1}^{N+1} \xi_i^k = \chi^H - x^k$ and $\xi_i^k \in \operatorname{dom} u_j(c_j, \cdot)$ for all $i, j, k$. Then, define $\theta^a$ such that $\theta^a(\xi_j^k) = 1/(N \cdot \hat{n})$ for all $k = 1, ..., \hat{n}, j = 1, ..., N-1$ and $\theta^a(\xi_N^k + \xi_{N+1}^k) = 1/(N \cdot \hat{n})$ for all $k = 1, ..., \hat{n}$. Similarly, define $\theta^1$ such that $\theta^1(\xi_j^k) = 1/((N+1) \cdot \hat{n})$ for all $k = 1, ..., \hat{n}, j = 1, ..., N+1$ and $\theta^0 = (\theta^1 + \theta^a)/2$. Clearly, the realizations of $\theta^a$, $\theta^0$ and $\theta^1$ are in the effective domain of every agent $i = 1, ..., N$. Notice that for any $a \in H$, there exists some $y \in \mathcal{B}^\perp$ such that $y^j = 0$, $\bar{w}_{N+1}^j \leq (\hat{n}-1)/\hat{n}$ and $\bar{\theta}^{a,j} = (1 - \bar{w}_{N+1}^j)/N$. Then, $\bar{\theta}^1 = N/(N+1) \cdot \bar{\theta}^a$ and $\bar{\theta}^0 = (2N+1)/(2N+2) \cdot \bar{\theta}^a$. This implies for each $j \in H$, $\bar{\theta}^{0,j} > \bar{\theta}^{1,j} > 0$.

Define the sequence of economies $\bar{\mathcal{E}}_m = \{(m \cdot u_i^1(c_i, \cdot), \theta^0, 1)_{i=1}^N, (u_{N+1}^0, w_{N+1})\}$ for $b_{N+1}$ large enough. We want to show there exists a strong equilibrium in $\bar{\mathcal{E}}_m$ for each $m$. Define $\bar{\mathcal{E}}'_m = \{(m \cdot u_i^1(c_i, \cdot), \theta^0, 1)_{i=1}^N, (u_{N+1}^0, b_{N+1})\}$, which differs from $\bar{\mathcal{E}}_m$ for agent $N+1$'s endowment. By Step 2, there exists a strong equilibrium $(p^m, \alpha^m)$ for each $\bar{\mathcal{E}}'_m$ and for all $b_{N+1}$. We claim that for $b_{N+1}$ large enough, agent $N+1$ will get the maximal utility $n^*$ in equilibrium. To see why this is true, notice that the total expenditure of agents 1 to $N$ is at most $N \cdot p^m \cdot \bar{\theta}^0 + N$, while the total expenditure on the market is $p^m \cdot \chi^H \leq N \cdot p^m \cdot \bar{\theta}^0 + N + b_{N+1}$. Set $b_{N+1}$ large enough respect to $N$, for any $\epsilon > 0$, we can make $(1-\epsilon)(\chi^H - N \cdot \bar{\theta}^0)$ affordable for agent $N$. Notice that $\chi^H - N \cdot \bar{\theta}^a = \bar{w}_{N+1}$ and $\bar{\theta}^0 = (2N+1)/(2N+2) \cdot \bar{\theta}^a$, there exists $\epsilon^* > 0$ such that $(1-\epsilon^*)(\chi^H - N \cdot \bar{\theta}^0) > \bar{w}_{N+1}$. Thus, for $b_{N+1}$ large enough, in the equilibrium, $w_{N+1}$ would be affordable for agent $N+1$ and then $u_{N+1}(\alpha^m) \geq u_{N+1}(w_{N+1}) = n^*$.

Now suppose $(p^m, \alpha^m)$ is a strong equilibrium for $\bar{\mathcal{E}}'_m$ with $u_{N+1}(\alpha^m) = n^*$ and $w_{N+1}$ is affordable for agent $N+1$, then we want to show that $(p^m, \alpha^m)$ is also a strong equilibrium for $\bar{\mathcal{E}}_m$. Notice that the only difference between the two economies is the endowment of agent $N+1$, it suffices to check the optimality of agent $N+1$. This is trivial as $(p^m, \alpha^m)$ is a strong equilibrium for $\bar{\mathcal{E}}'_m$ and $w_{N+1}$ is affordable.

*Step 4:* The rest of the proof is analogous to the arguments given in the proof of Theorem 3. Here we only point out the required adjustments.

If there exists a bounded subsequence of $p^m$, the argument is unchanged once we observe that agent $N+1$ gets the maximal utility at each point in the sequence and, therefore, also in the limit.

If $\sum_{j=1}^L p^{m,j} \to \infty$, since $\bar{\theta}^{0,j} > 0$, we know $p^m \cdot \bar{\theta}^0 \to \infty$. Then, we define $\hat{p}^m = p^m/(p^m \cdot \bar{\theta}^0) \in [0, 2N \cdot \hat{n}]^L$ and let $(p, \alpha)$ be the limit point of a convergent subsequence of $(\hat{p}^m, \alpha^m)$. We must verify that $p \cdot \bar{\alpha}_i = 1$ for all $i = 1, ..., N$. By construction, $\hat{p}^m \cdot \bar{\theta}^0 = 1$. Therefore,

$$\sum_{i=1}^N \hat{p}^m \cdot \bar{\alpha}_i^m + \hat{p}^m \cdot \bar{\alpha}_{N+1}^m \geq N \cdot \hat{p}^m \cdot \bar{\theta}^0 + \hat{p}^m \cdot \bar{w}_{N+1} = N + \hat{p}^m \cdot \bar{w}_{N+1}$$

By definition of a strong equilibrium, $\hat{p}^m \cdot \bar{\alpha}_{N+1}^m \leq \hat{p}^m \cdot \bar{w}_{N+1}$ and thus $\sum_{i=1}^N \hat{p}^m \cdot \bar{\alpha}_i^m \geq N$. Passing to limit, $\sum_{i=1}^N p \cdot \bar{\alpha}_i \geq N$. A standard argument now implies that $p \cdot \bar{\alpha}_i \leq 1$ for all $i = 1, ..., N$. Thus, $p \cdot \bar{\alpha}_i = 1$ for all $i = 1, ..., N$. Now, we can repeat the corresponding argument in Theorem 3 to show that there exists a strong equilibrium in $\hat{\mathcal{E}}_c$. □

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