IO in I-O: Size, Industrial Organization, and the Input-Output Network Make a Firm Structurally Important *

Basile Grassi†

Bocconi University and IGIER

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Abstract

Firm-level productivity shocks can help understand sector- and macroeconomic-level outcomes. Capturing the market power of these firms is important: it determines how productivity gains translate into prices and markups. In existing models, firms do not internalize the impact of their systemic size. This paper explores the alternative oligopolistic market structure. To this end, I build a tractable multi-sector heterogeneous-firm general equilibrium model featuring oligopolistic competition and an input-output (I-O) network. By affecting price and markup, firm-level productivity shocks propagate both to the downstream and upstream sectors. Sector-level competition intensity affects the strength of these new propagation mechanisms. The structural importance of a firm is determined by the interaction of (i) the sector-level competition intensity, (ii) the firm's sector position in the I-O network, and (iii) the firm size. In a calibration exercise, the aggregate volatility arising from independent firm-level shocks is 34% of the one observed in the data.

Keywords: Input-Output Network, Production Network, Shocks Propagation, Oligopoly, Imperfect Competition, Industrial Organization, Firm Heterogeneity, Random Growth, Granularity, Volatility, Micro-Origin of Aggregate Fluctuations, Business Cycle.

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†Bocconi University, Department of Economics, Via Rontgen 1, 20136 Milano, Italy. Email: basile.grassi@unibocconi.it

1 Introduction

Firm-level productivity shocks can explain an important part of movement in prices and output at the sector and macroeconomic level.[1] The market power of these firms in their sector, that is, the market structure, determines how productivity gains translate into prices and markups, and, thus, how they affect the economy. Imperfect competition appears to describe many markets more accurately than perfect competition: a variety of market structures exist. But how important is the variety of market structures for sector- and macroeconomics-level variables? In particular, how important is the variety of market structures for understanding the propagation of firm-level productivity shocks through the input-output (I-O) network? This is the question analyzed in this paper. The idea behind the micro-origin of aggregate fluctuations is that a handful of large firms represent a large share of a sector, and thus shocks hitting these large firms cannot be balanced out by the shocks affecting smaller firms. However, the typical model is restrictive regarding the nature of competition within a sector: firms are large enough to have a systemic importance, but these firms do not internalize it when they make their decisions. For example, a firm whose sales represent 50% of its market behaves as if prices were given. This paper explores the alternative oligopolistic market structure in which firms do take into account the effect of their decisions on sector-level price and quantity. The propagation of firm-level shocks to sector- and aggregate-level variables is shown to depend on the competition intensity, the I-O network, and the firm size.

Table 1 and Figure 1 motivate this paper: sectors are concentrated and linked through a "small-world" I-O network. Table 1 shows summary statistics of the top four firms' share of industry revenue in 2002, 2007, and 2012 for around 970 industries. Industry revenue accounted for by the top four firms varies from almost zero to close to 100%, with a median value close to 33% in 2007. The first thing to note is that large firms represent an important share of revenue of the median sector. Second, because concentration is a widely used measure of a sector's competition intensity, this table also suggests different sectors have different competition levels. For the bottom 25% of these sectors, the top four firms account for less than 18% of the total industry revenues, whereas for the top 25% of these sectors, only four firms account for more than 50% of the total industry revenues. While confirming the "granular" nature of these sectors, this table emphasizes the heterogeneity across sectors of the intensity of competition. In addition, these sectors are not independent from each other: production in one sector relies on a complex and interlocking supply chain. Figure 1 displays the I-O network among 389 sectors for the United States in 2007. This is a "small-world" network: a few nodes are connected to many other nodes. In such production networks, as Acemoglu et al. (2012) and Carvalho (2010) show, these highly-connected sectors propagate sector-level fluctuations, which translate into sizeable aggregate fluctuations.

[1] An important paper in the literature on the micro-origin of aggregate fluctuations is the seminal work by Gabaix (2011) that shows that when the firm-size distribution is fat-tailed, firm-level shocks do not wash out at the aggregate. Building on this seminal work, Carvalho and Grassi (2017) show that firm dynamic models contain a theory of business cycles as soon as the continuum of firms' assumption is relaxed. Acemoglu et al. (2012), Carvalho (2010, 2014), and Baqaee (2016) build on the multi-sector business-cycle framework of Long and Plosser (1983) to show how shocks on sectors linked through an I-O network can translate into aggregate fluctuations. Earlier contributions include Jovanovic (1987), Durlauf (1993), and Bak et al. (1993).

Table 1: Top 4 firms' share of total industry revenues

Year	Mean	Q1	Q2	Q3	Std
2002	35.4	17.6	31.1	51.0	23.0
2007	37.2	18	32.9	53.2	23.8
2012	35.0	16.2	31.2	50.1	22.9

NOTE: Summary statistics of the distribution of the top four firms' share of total industry revenues across 970 industries. The second column is the unweighted mean, the third column is the first quartile, the fourth column is the median, the fifth column is the third quartile, and the sixth column is the standard deviation. Source: US Census Bureau, 6-digit NAICS industries, all sectors except 11 , 21, 23, 55 , and 92.

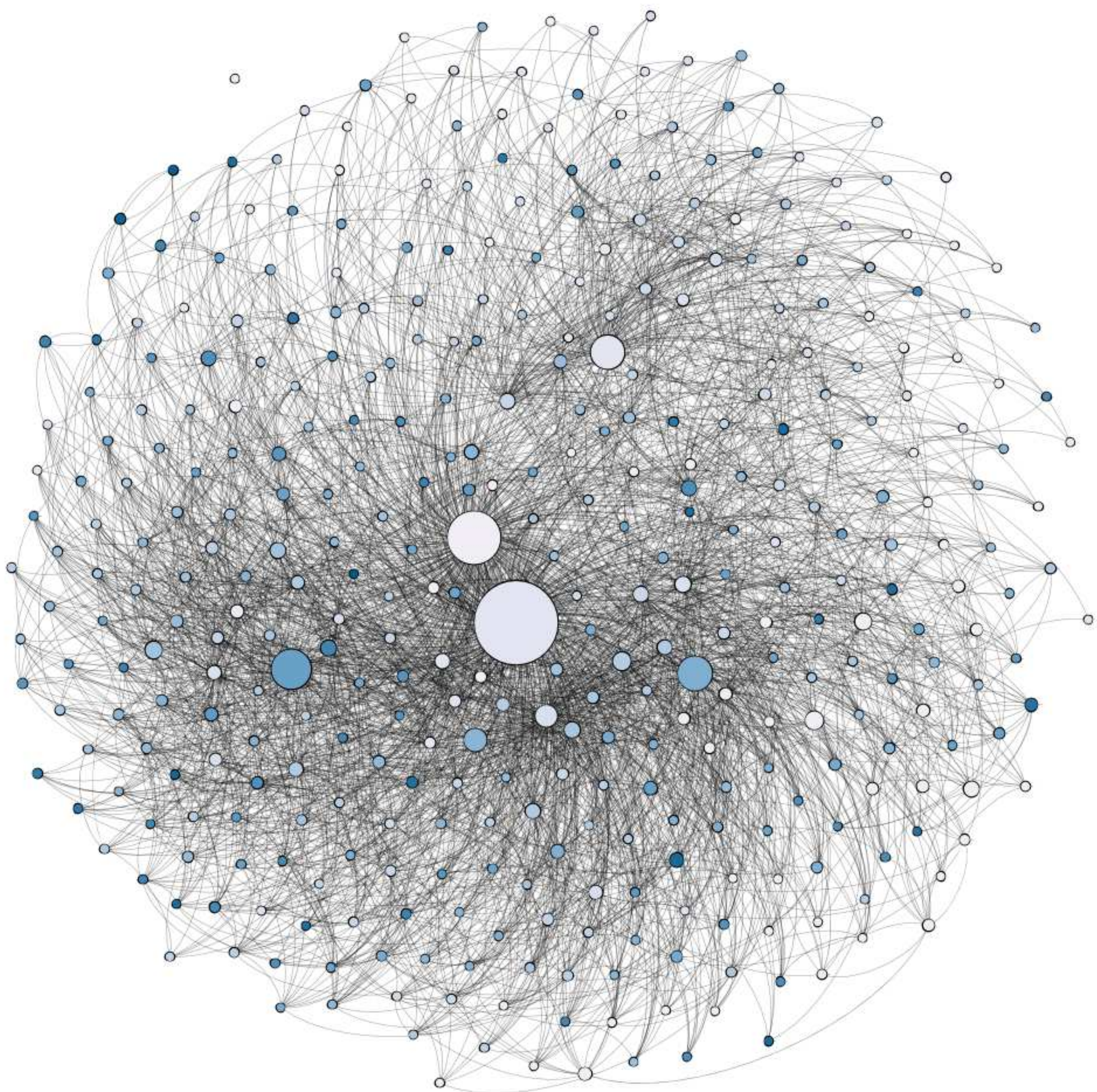
In this paper, I characterize how the structural importance of a firm depends on the interaction between the competition intensity, the I-O network, and the firm size.[2] Furthermore, I show how the intensity of competition affects the propagation of firm-level shocks in the I-O network. Finally, I quantify how much aggregate volatility can be generated by independent firm-level shocks. To this end, I build a tractable multi-sector heterogeneous-firm general equilibrium model featuring oligopolistic competition and an I-O network. Within each sector, a finite number of heterogeneous firms are subject to oligopolistic competition and set variable markups à la Atkeson and Burstein (2008). These firms are subject to persistent idiosyncratic (labor-augmenting) productivity shocks that translate into sector-level and aggregate fluctuations.

The mechanism by which a firm affects sector- and aggregate-level output is as follows. Consider a sector with a finite number of heterogeneous firms and assume that an already-large firm is subject to a positive productivity shock. Following this shock, the sector's average productivity increases because the productivity of one firm has increased. Because this firm was already large before the shock hit, the sector becomes even more concentrated. This firm-level shock has two opposite effects on price and output at the sector level. First, because of the increase in average productivity, the sector good is cheaper and output increases. Second, because of the increase in concentration, competition in the sector decreases: this large firm is larger and can use its size to extract even more profit and charge a higher markup. It follows that the reduction in competition intensity mitigates the decrease in price and increase in output.

These changes in prices and output propagate to the other sectors via the I-O network through two channels. First, the fall in price reduces the marginal cost of downstream sectors. Indeed, the downstream sectors use this good as an input to produce. These downstream sectors also reduce their price, and the downstream of these downstream sectors see their marginal cost fall. Ultimately, every downstream sector reduces its price, which results in an increase in the real wage. Because the decrease in competition intensity mitigates the initial fall in price, it also determines the strength of this downstream propagation mechanism. Second, the decrease in competition also propagates to the upstream sectors. Indeed, lower competition increases the sector's profit share and reduces its input share. It follows that the demand faced by upstream sectors falls. But, these upstream sec-

[2] The structural importance of a firm is, here, the elasticity of aggregate output to the productivity of this firm.

Figure 1: The US I-O Network in 2007



NOTE: Larger nodes of the network represent sectors supplying inputs to many other sectors. A darker color represents the higher top four firms' share of total revenues in 2007 (sectors without available data are left white). The figure represents 389 sectors. Source: Bureau of Economic Analysis, detailed I-O table for 2007 and Census Bureau. The figure is drawn with the software package Gephi.

tors are also reducing the demand for their inputs. In turn, the demand faced by the suppliers of these upstream sectors is also reduced. This upstream propagation mechanism ultimately affects the repartition of income across sectors, and between profit and labor income.

Through these downstream and upstream propagation mechanisms, the increase in productivity of one firm in one sector affects the real wage, the repartition of income between labor and profit, and the aggregate output. Note that the above intuition traces the particular effect of a shock on an already-large firm in one sector. In this paper, a large integer number of firms are subject to persistent independent productivity shocks across a large integer number of heterogeneous sectors. These shocks translate into non-negligible sector- and aggregate-level fluctuations through the downstream and upstream propagation mechanisms. Because large firms are disproportionaly important, the diversification across firms' shocks in a sector is weak: the "granular hypothesis" of Gabaix (2011) is at play. The diversification across sectors is dampened because of the "small-world" nature of the I-O network as shown by Acemoglu et al. (2012) and Carvalho (2010).

In this model, up to an approximation, only two sufficient statistics at the sector level are needed to solve for the equilibrium allocation analytically. For each sector, the first statistic is the cross-sectional sum of firm-level productivity, whereas the second statistic is the cross-sectional Herfindahl index of firm-level productivity. The high degree of tractability allows me to compute the elasticity of sector- and aggregate-level prices and income shares.

I show the effect of the change in productivity of a firm in a given sector on aggregate output is a function of four characteristics. First is the sector's concentration, which determines the competition intensity in that sector and, thus, how much a shock to a firm translates into a change in sector-level markup and price. Second is the sector centrality, which measures that sector's direct and indirect importance in the household's consumption bundle. This characteristic relates to the transmission of firm-level shocks to downstream sectors. Third is the sector's profit share over its whole supply chain, which measures the profits that the sector captures directly and indirectly (through the I-O network). This characteristic relates to the propagation of firm-level shocks to upstream sectors. The fourth characteristic is the firm size, which interacts with all characteristics, and determines the strength of the downstream and upstream propagation.

Furthermore, I show that a change in a firm's productivity in a given sector propagates to the price of downstream sectors and to the sales share of upstream sectors. Because of oligopolistic competition, a change in one firm's productivity does not pass through fully on sector-level price, but it affects the profit and cost share. The change in price propagates to downstream sectors but is either reduced or magnified, relative to the full-pass-through case, depending on the competition intensity and the identity of the firm subject to the shock. Because the change in profit share affects the demand for inputs, it propagates to upstream sectors and affects their sales as a share of output. The sign of this upstream propagation mechanism is jointly determined by the competition intensity and the identity of the firm. The latter mechanism requires both oligopolistic competition and an I-O network.

Thanks to the high tractability of the model and the fact that the equilibrium is characterized by only two sector-level sufficient statistics, I can calibrate this economy by relying only on the US Census Bureau' concentration data which pin down sectors' competition intensity, and the Bureau of Economic Analysis (BEA)'s I-O data. For the benchmark calibration, the output volatility that comes out of simulation is 34% of what is observed in the data. In the model, the volatility of the labor-income share is about 11% of the aggregate volatility, whereas the volatility in wage due to the change in competition intensity represents almost 16% of aggregate volatility. Furthermore, the effect of competition on sector-level volatility is heterogeneous across sectors.

Related Literature: This paper contributes to the literature on the micro-origin of aggregate fluctuations. This literature is based on two mains ideas: the "granular hypothesis" and the network origin. For the former, seminal work by Gabaix (2011) shows that whenever the firm-size distribution is fat-tailed, idiosyncratic shocks do not average out quickly enough and therefore translate into sizable aggregate fluctuations. Carvalho and Grassi (2017) ground the "granular hypothesis" in a well-specified firm-dynamic setup. For the latter, Acemoglu et al. (2012) and Carvalho (2010) show that when the distribution of sectors' centrality in the I-O network is fat-tailed, sector-level perturbations also generate sizable aggregate fluctuations. Relative to these papers, I present the first framework that includes both components explicitly. The "granular hypothesis" leads to sector-level fluctuations, whereas the I-O network structure translates sector-level fluctuations into aggregate fluctuations.[3] An important drawback of this literature is that firms are assumed to be large enough to influence the aggregate but also small enough not to be strategic. Here, I present the first model of strategic pricing, in which aggregate fluctuations arise from purely idiosyncratic shocks. When they are making their decisions, firms do take into account the fact that they have market power and can influence their sector's output and price.

Recently, Baqaee and Farhi (2017a) revisited the famous and influential result by Hulten (1978) that, in efficient economies, the first-order impact of a productivity shock to a firm on aggregate output is equal to that firm's sales as a share of output. The framework presented here is not subject to this result, because the economy is not efficient. Therefore, it is closer to Basu (1995), Basu and Fernald (2002), Jones (2011, 2013), Bigio and La'O (2016), or Baqaee and Farhi (2017b), who study the introduction of distortions in a multi-sector macroeconomic model with production networks. In contrast to all of these papers, here, firm-level productivity shocks endogenously affect markups, which generate the distortions in this economy.

An important literature studies the transmission of shocks across sectors through the I-O network: Acemoglu et al. (2015) look at the transmission of well-identified supply and demand shocks; Carvalho et al. (2016) and Boehm et al. (2016) study the firm-level impact of supply-chain disruptions occurring in the aftermath of the Great East Japan Earthquake in 2011, and Barrot and Sauvagnat (2016) look at the effect of natural disasters. Baqaee (2016) studies theoretically the effect of shocks on entry cost. In this paper, I introduce a new propagation mechanism of firm-level shocks in the I-O network through changes in sector-level competition, which act as supply shocks to downstream

[3]Notable contribution, in this literature include but are not limited to di Giovanni et al. (2014), Magerman et al. (2016), and Baqaee and Farhi (2017a).

sectors and demand shocks to upstream sectors. Firm-level shocks propagate both downstream and upstream despite the Cobb-Douglas assumption; furthermore, firm size and the sector-level competition intensity jointly determine the strength of the downstream propation and the sign of the upstream propagation.

This paper also contributes to the literature on imperfect competition among heterogeneous firms. Krugman (1979), Ottaviano et al. (2002), Melitz and Ottaviano (2008), Bilbiie et al. (2012), and Zhelobodko et al. (2012) study demand-side pricing complementarities, whereas I study at supply-side pricing complementarities as in Atkeson and Burstein (2008) but in an I-O context. Furthermore, I show, up to an approximation, that such a model is highly tractable and that firm heterogeneity can be summarized at the sector level by just two sufficient statistics.

Finally, this paper relates to a recent and growing empirical literature that documents and analyzes the macroeconomic consequences of the rise in market concentration in the United States. Barkai (2017), Autor et al. (2017), and Kehrig and Vincent (2017) explain the secular declines in labor share from the increase in sector-level and firm-level concentration, whereas Loecker and Eeckhout (2017) document an increase in firm-level markup, which they relate to a number of secular trends in the last three decades. Even if the focus of this paper is different, it contributes to this literature by providing a simple and tractable model to analyze the aggregate consequence of market concentration. For example, market concentration is shown here to be driving sector-level markup and therefore the profit and labor share.

Outline: The paper is organized as follows. In Section 2, I describe and solve the household's and firm's problem. In Section 3, I first aggregate firms' behavior at the sector level and show that firm heterogeneity can be summarized by two sufficient statistics. I then solve for the dynamics of these two statistics. In Section 4, I show that a firm's sector market structure, its role in the I-O network, and the firm's size jointly determine its structural importance. In Section 5, I look at how firm-level productivity shocks propagate to other sectors through the I-O network. In Section 6, I calibrate the model and perform some quantitative exercises. Finally, Section 7 concludes.

2 Model

In this section, I describe the structure of the economy and solve for the household and firms' problem. Two types of agent exist. First, a representative household consumes and supplies labor. Second, a finite number of firms are distributed across a finite number of sectors that are linked by a production network. In each sector, firms set their price (or quantity) strategically. Each firm is subject to independent persistent idiosyncratic shocks.

2.1 Household

The representative household lives a discrete and infinite number of periods. Preferences are given by $\mathbb{E}_0 \sum_{t=0}^{\infty} \rho^t u(C_t, L_t)$, where $u(C_t, L_t)$ is the instantaneous utility, ρ is the discounted rate, C_t is the composite consumption good, and, L_t is the number of hours worked at time t.

The composite consumption good C_t is a Cobb-Douglas aggregation of $N \in \mathbb{N}$ sector-level goods: $C_t = \theta \prod_{k=1}^{N} C_{k,t}^{\beta_k}$, where $C_{k,t}$ is the amount of good k consumed by the household at time t and where θ is a normalization constant.[4] The Cobb-Douglas weights, β_k, are equal to the expenditure shares of each good $\frac{P_{k,t}C_{k,t}}{P_t^C C_t}$, where $P_{k,t}$ is the price of good k and P_t^C is the aggregate price index that satisfies $P_t^C = \prod_{k=1}^{N} P_{k,t}^{\beta_k}$. Note that N is an integer number.

In a sector k, an integer number, N_k, of varieties exists indexed by i. These varieties are aggregated with a constant elasticity of substitution $\varepsilon_k > 1$ such that $C_{k,t} = \left(\sum_{i=0}^{N_k} C_t(k,i)^{\frac{\varepsilon_k - 1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}}$, where $C_t(k,i)$ is the amount of sector k's variety i consumed by the household at time t. Finally, the price of good k satisfies $P_{k,t} = \left(\sum_{i=1}^{N_k} P_t(k,i)^{1-\varepsilon_k}\right)^{\frac{1}{1-\varepsilon_k}}$, where $P_t(k,i)$ is the price of variety i in sector k at time t. Each variety is produced by exactly one firm, and all the firms are owned by the representative household.

The above household preferences and the assumption that $\varepsilon_k > 1$ capture the idea that as one is dissagregating further, from sectors to firms, the household can more easily substitute between two dissagregating units. Furthermore, the degree of substitution between two varieties of the same good is higher that between two varieties of two different goods.

2.2 Firms

An integer number of firms are split into N sectors. Sector k contains N_k firms and each variety is produced by exactly one firm. Firms are heterogeneous in their (labor-augmenting) productivity. A sector is defined as a technology and a market structure: (i) firms in the same sector have access to the same production function and (ii) these firms compete with each other in a differentiated Bertrand or Cournot game. At the end of this section, I show the implied firm dynamics in this model are consistent with recent empirical evidence.

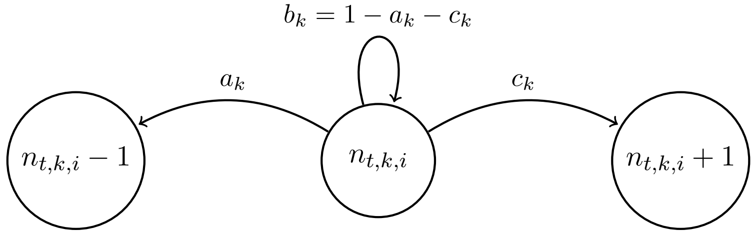
Technology: The firm i in sector k combines labor, $L_t(k,i)$, and other sectors' goods, $x_t(k,i,l)$, to produce $y_t(k,i)$ units of its variety using the constant return-to-scale Cobb-Douglas technology $y_t(k,i) = \alpha_k \Big(Z_t(k,i)L_t(k,i)\Big)^{\gamma_k} \prod_{l=1}^{N} x_t(k,i,l)^{\omega_{k,l}}$, where γ_k is the labor share in the production, α_k is a normalization constant,[5] $Z_t(k,i)$ is the labor-augmenting productivity specific to the firm i in sector k, and $\omega_{k,l}$ is the input share of sector l's goods needed in sector k's production. The $(N \times N)$ matrix $\Omega = \{\omega_{k,l}\}_{k,l}$ represents the I-O network.[6] Thanks to constant return-to-scale, the kth rows of Ω sum to $1-\gamma_k$: $\sum_{l=1}^{N} \omega_{k,l} = 1-\gamma_k$. Furthermore, $x_t(k,i,l)$ is a composite of sector l's varieties such that $x_t(k,i,l) = \left(\sum_{j=1}^{N_l} x_t(k,i,l,j)^{\frac{\varepsilon_l - 1}{\varepsilon_l}}\right)^{\frac{\varepsilon_l}{\varepsilon_l - 1}}$, where $x_t(k,i,l,j)$ is the quantity of the variety j of sector l's good that is used for the production of variety i of sector k's good. Note the elasticity of substitution among varieties in a sector is the same for firms and for the household.

[4]The normalization constant θ makes the mathematics simpler and is equal to $\theta = \prod_{k=1}^{N} \beta_k^{-\beta_k}$.

[5]The normalization constant α_k makes the mathematics simpler and is equal to $\alpha_k = \gamma_k^{-\gamma_k} \prod_{l=1}^{N} \omega_{k,l}^{-\omega_{k,l}}$.

[6]The notation $U = \{u_{k,l}\}_{k,l}$ means U is the matrix where the element k,l is equal to $u_{k,l}$, whereas I denote $v = \{v_k\}_k$ as the vector where the element k is equal to v_k.

Figure 2: Productivity Process



NOTE: A representation of the transition probabilities in assumption 1 of a firm i in sector k for $M_k > n_{t,k,i} > 0$.

The I-O network Ω is assumed to be fixed across time and state, because it is a sector-level network. Here the I-O linkages are interpreted as technology: the input bundle needed to produce the variety of a good. At the business-cycle frequency, the labor-augmenting firm-level productivity shocks that are considered here do not affect this technology. However, if one firm in a sector significantly increases the price of its variety, its customers are able to substitute away from this variety thanks to the double-nested constant-elasticity-of-substitution demand system. Therefore, even if the sector-level I-O linkages are fixed, the transaction network between firms is not and varies across time and state.[7]

The productivity of firm i in sector k, $Z_t(k,i)$, is identically and independently distributed across firms but not across time. It follows a sector-specific Markov chain over the discrete state space, $\Phi_k = \{1, \varphi_k, \varphi_k^2, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\} = \{\varphi_k^n\}_{n \in \{0,1,\ldots,M_k\}}$ for $\varphi_k > 1$, which is evenly distributed in logs.[8] This Markov chain is described by the matrix of transition probabilities $\mathcal{P}^{(k)} = \{\mathcal{P}^{(k)}_{n,n'}\}_{n,n'}$ where $\mathcal{P}^{(k)}_{n,n'} = \mathbb{P}\left(Z_{t+1}(k,i) = \varphi_k^{n'} | Z_t(k,i) = \varphi_k^n\right)$ is the probability that a firm i in sector k jumps from productivity level φ_k^n to $\varphi_k^{n'}$ between time t and time $t+1$. In some cases, I assume a specific Markovian chain, which is a discretization of a random growth process and is taken from Córdoba (2008). Figure 2 and Assumption 1 describe its transition probabilities.

Assumption 1 ***(Random Growth)*** *For $a_k + b_k + c_k = 1$, firm-level productivity in sector k follows a Markov chain over $\Phi_k = \{\varphi_k^n\}_{n \in \{0,1,\ldots,M_k\}}$ with transition probabilities such that:*

$$\mathcal{P}^{(k)} = \begin{pmatrix} a_k + b_k & c_k & 0 & \cdots & \cdots & 0 & 0 \\ a_k & b_k & c_k & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k & b_k & c_k \\ 0 & 0 & 0 & \cdots & 0 & a_k & b_k + c_k \end{pmatrix}$$

Pricing: A sector is also defined as a market in which firms are engaged in imperfect competition. Sector's goods are imperfect substitutes and varieties within a sector are more substitutable: $\varepsilon_k > 1$. Each firm produces exactly one variety of its sector's good, and customers cannot perfectly substitute

[7]This assumption differs from Magerman et al. (2016) who study the micro-origin of firm-level shocks in a fixed firm-to-firm transaction network.

[8]This assumption implies $\varphi_k^{n+1}/\varphi_k^n = \varphi_k$.

between two varieties: $\varepsilon_k < \infty$. Following Atkeson and Burstein (2008), I assume firms play a static game where firm i in sector k chooses its price $P_t(k,i)$ taking as given the prices chosen by other firms in the economy, the other sectors' price and quantities, the wage, and, aggregate prices and quantities. Importantly, note that this firm recognizes that sector k's price and quantity are affected when it changes its price.

To understand this assumption, imagine General Motors (GM) has a way to produce a car that costs $10,000 less than its competitors. The above assumption implies that when GM is making its pricing decision, it is internalizing the impact of its decision on the quantity and price of the "Automobile Manufacturing" sector but not on the "Amusement Parks and Arcades" sector. Note that with these assumptions in place, GM is not internalizing the impact of its pricing decision on the real wage and on the prices and quantities of its upstream or downstream sectors. Relaxing these assumptions might create effects that will go beyond the scope of this paper, and I leave these questions for future research.[9]

Note that here I assume a competition in price (labeled as *Bertrand*). In most of the results below, I compare the baseline case of *Bertrand* competition with (i) the case of *Cournot* competition where firms compete in quantity and (ii) with the benchmark case of *monopolistic* Dixit and Stiglitz (1977) competition. When doing so does not create confusion, I abstract from the time t subscript.

As a result of cost minimization, firm i in sector k faces a marginal cost $\lambda(k,i) = Z(k,i)^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$, where w is the wage rate in this economy. Note that due to the presence of I-O linkages, this marginal cost is a function of other sectors' prices. The sector-level gross output is defined as $Y_k = \left(\sum_i^{N_k} y(k,i)^{\frac{\varepsilon_k - 1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}}$. Proposition 1 characterizes the pricing decision of a firm i in sector k.

Proposition 1 *(**Firm's Pricing**) Firm i in sector k sets a price $P(k,i)$, a markup $\mu(k,i)$ and has a sale share $s(k,i)$ that satisfies the following system of equations:*

$$
\begin{aligned}
P(k,i) &= \mu(k,i)\lambda(k,i) \\
s(k,i) &= \frac{P(k,i)y(k,i)}{P_k Y_k} = \left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k} \\
\mu(k,i) &= \begin{cases} \frac{\varepsilon_k}{\varepsilon_k - 1} & \textit{Under Monopolistic Competition} \\ \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} & \textit{Under Bertrand Competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} & \textit{Under Cournot Competition.} \end{cases}
\end{aligned}
$$

Proof *See Atkeson and Burstein (2008).* □

The first thing to note in the above proposition is that firms charge a markup over their marginal cost. Under monopolistic Dixit and Stiglitz (1977) competition, the markup is constant and equal to $\varepsilon_k/(\varepsilon_k - 1)$. Under oligopolistic competition (Bertrand or Cournot), the markup charged is increasing in the sales share of the firm: larger firms charge a higher markup.

[9] Another interpretation of the assumptions made here is that firms have limited ability to compute the effect of their decision on any variable outside their sector's price and quantity. This assumption is fundamentally different from the Atkeson and Burstein (2008) framework because, in their paper, they assume a continuum of sectors: even if firms are not atomistic within a sector, a firm's sector is atomistic with respect to the aggregate economy.

Note that in both the Bertrand and Cournot competition case, the markup a firm charges is converging to a constant as the size of this firm goes to zero. Indeed, for firm i in sector k, we have $\mu(k,i) \to \varepsilon_k/(\varepsilon_k - 1)$ as $s(k,i) \to 0$. As a firm becomes atomistic, its markup approaches the one under monopolistic competition. Because the system of equation in Proposition 1 does not admit an analytical solution, aggregating firms' behavior at the sector level in a tractable way turns out to be impossible. To circumvent this issue, Proposition 2 is approximating the sales share of a firm under oligopolistic competition by the sales share of this firm under monopolistic competition. In Section 3, this result is used to collapse firms' heterogeneity to two sector-level statistics.

Proposition 2 *(**Firm's Approximation**) The sales share of firm i in sector k under monopolistic competition is a function of its marginal cost $\lambda(k,i)$ and the sector k price index: $\widehat{s}(k,i) = \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$. When $\widehat{s}(k,i) \to 0$, the sales share of this firm under oligopolistic competition, $s(k,i)$, satisfies*

$$s(k,i) = \begin{cases} \widehat{s}(k,i) - \left(1 - \varepsilon_k^{-1}\right)\widehat{s}(k,i)^2 + o\Big(\widehat{s}(k,i)^2\Big) & \textit{Under Bertrand Competition} \\ \widehat{s}(k,i) - \left(\varepsilon_k - 1\right)\widehat{s}(k,i)^2 + o\Big(\widehat{s}(k,i)^2\Big) & \textit{Under Cournot Competition.} \end{cases}$$

where the notation $f(x) = o\Big(g(x)\Big)$ means $f(x)/g(x) \to 0$ when $x \to 0$.

Proof *See Appendix A.1.* □

In this proposition, the sales share of firm i in sector k is approximated by the sales share under monopolistic competition $\widehat{s}(k,i)$. Because a one-to-one mapping exists between the marginal cost and $\widehat{s}(k,i)$ for a fixed sector price index P_k, one can think of this result as an approximation of the sales share of firms as a simple function of their marginal cost. Similarly, the above results hold when $\widehat{s}(k,i)$ is small or, because $\varepsilon_k > 1$, when the marginal cost $\lambda(k,i)$ is large.

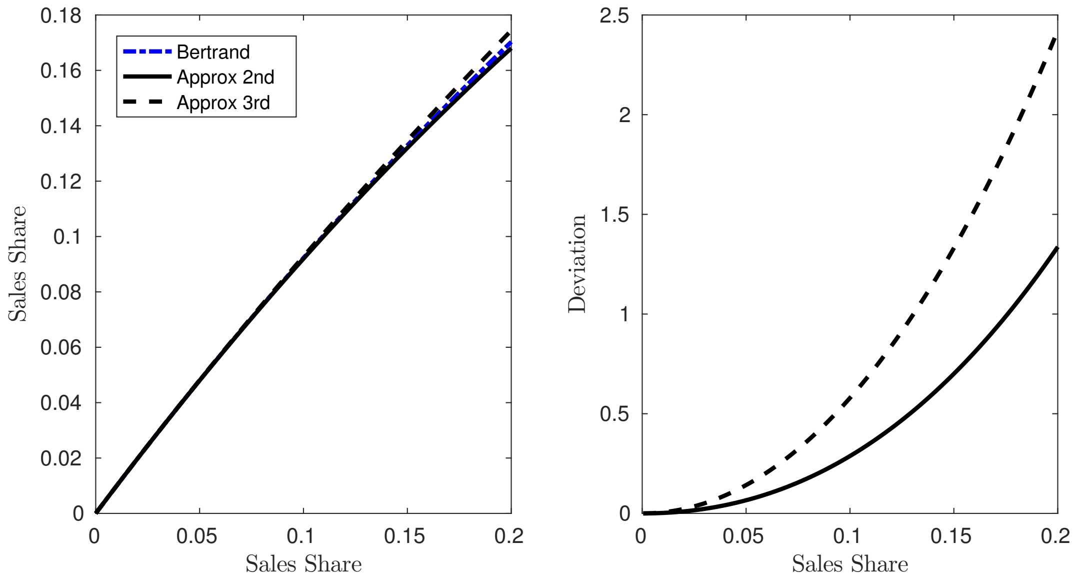
The framework derived in this paper is designed to capture the aggregate effect of shocks on "large" firms, whereas the results in Proposition 2 hold for "small" firms in their market. Therefore, knowing if "small" in the sense of the approximation in Proposition 2 is "small" economically is important. Figure 3 displays on the left panel the sales share under Bertrand competition as a function of the sales share under monopolistic competition along with the second- and third-order approximation. The right panel of this figure displays percentage deviations of the approximations with respect to the exact solution. For sales share up to 20%, the error made by the second-order approximation is less than 1.5%: "small" for the approximation in Proposition 2 is thus not small economically. To aggregate the firms' behavior at the sector level, in some cases, I assume this approximation holds. [10, 11]

Assumption 2 ***(Approximation)** In Proposition 2's approximation, higher-order terms are negligible.*

[10] For conciseness, the third-order approximation is not reported in the formula in Proposition 2 but can be found in the proof of this proposition in Appendix A.1.

[11] A concern could be that this approximation holds well only in levels and not in term of slopes. Figure 10 in Online Appendix E shows that for sales share up to 20%, the error made on the slope is less than 5%. Another concern might be that the quality of this approximation depends on the value of the elasticity across varieties in a sector ε_k. Figures 11 and **??** in Online Appendix E show the quality of the approximation is of the same order for different values of ε_k.

Figure 3: Approximation of Firms' Sales Share



NOTE: For $\varepsilon_k = 5$. The left panel shows the Bertrand sales share using a numerical solver, the second- and the third-order approximation as a function of the monopolistic sales share. The right panel shows the percentage deviation of both approximations with respect to the numerical solution.

Firm Dynamics: To conclude the description of the model, let us look at its implication in term of firm dynamics. Under Assumption 1, the productivity of a given firm satisfies Gibrat's law: the growth rate is independent of the level. In particular, the level of productivity does not affect the mean and the volatility of its growth rate. However, here Gibrat's law is violated for a firm's sales: the larger a firm, the larger its market power and the less sensitive its sales are to a change in its marginal cost. Indeed, the firm-level markup adjusts, and thus the pass-through of shock to price is incomplete. Proposition 3 formalizes this reasoning.[12]

Proposition 3 *(**Size-Volatility**) Under Assumptions 1 and 2, the (conditional) variance of the growth rate of firm i in sector k's productivity and sales share satisfies:*

$$\mathbb{V}ar_t\left[\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)}\right] = \sigma_k^2 \qquad \textit{and} \qquad \mathbb{V}ar_t\left[\frac{s_{t+1}(k,i) - s_t(k,i)}{s_t(k,i)}\right] = g_k(\hat{s}_t(k,i))\sigma_k^2$$

where $\sigma_k^2 = a\left(\varphi_k^{-1} - 1\right)^2 + c\left(\varphi_k - 1\right)^2 - \left(a\varphi_k^{-1} + b + c\varphi_k - 1\right)^2$ and $g_k : x \mapsto g_k(x)$ is a decreasing function. Furthermore, the slope of g_k is increasing in ε_k.

Proof *See Online Appendix F.1.* □

The fact that larger firms tend to be less volatile is well established. Recently, Yeh (2017) explored empirically the possible mechanisms that could give rise to such a negative relationship between size and volatility at the firm level. After ruling out diversification among establishments or products, he concludes that large firms face smaller price elasticities and therefore respond less to a given-sized

[12]Gibrat's law was first introduce by Gibrat (1931). See also Sutton (1997) for a review.

productivity shock than small firms do. In the current framework, the reason behind the negative size-volatility relationship of Proposition 3 is exactly the one identified by Yeh (2017).[13]

The simple demand system and the market structure assumed here, together with the random growth process for productivity implies, a rich and empirically relevant firm dynamic. As the rest of this paper shows, non-negligible sector-level and aggregate fluctuations arise from firm-level productivity shocks despite the fact that larger firms are less volatile.

3 Sector Aggregation

The model derived above describes an economy in which a finite number of firms, subject to productivity shocks, evolve and compete in their sector. The behavior and the dynamics of these firms shape the sector-level variables. This section characterizes the mapping between firm-level and sector-level variables. It shows the latter are related to a few moments of the distribution of firms whose dynamics are solved for.

This section is organized as follow. First, I introduce two key sector-level statistics. Second, I derive the relationship between sector-level markup and concentration before describing the equilibrium under Assumption 2. Finally, I describe the sector dynamics under Assumptions 1 and 2.

3.1 Two Statistics

In the model described in Section 2, given the distribution of productivity $Z_t(k,i)$ at each time t, one can solve for the equilibrium allocation. The distributions of productivity in each sector are the state variables of this economy. I introduce two moments of these distributions that turn out to be key to describe the equilibrium allocation under Assumption 2.

For a given sector k, the first statistic is the sum of the productivity of sector k's firms raised at a power that takes into account the downward-sloping demand and the decreasing return in labor:

$$\overline{Z_{t,k}} = \sum_{i=1}^{N_k} Z_t(k,i)^{(\varepsilon_k - 1)\gamma_k}.$$

This statistic is proportional to the unweighted average of firm-level productivity (raised at a power) in sector k and is therefore related to the first moment of the firm's productivity distribution in that sector. Note that sector k contains an integer number of firms N_k; therefore, when the productivity of one firm changes, this finite sum of productivity changes. More precisely, the elasticity of $\overline{Z_{t,k}}$ with respect to the productivity of firm i in sector k is $\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} = (\varepsilon_k - 1)\gamma_k \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} > 0$. If a continuum of firms rather than an integer number of firms existed then this elasticity would always be zero.

[13] Empirical studies that identified a negative size-volatility relationship at the firm level are Comin and Philippon (2006) and Comin and Mulani (2006) for publicly listed US firms, Fort et al. (2013) and Foster et al. (2008) for US manufacturing firms, and di Giovanni et al. (2014) using a census of French firms.

The second statistic is related to the second moment of the firms' productivity distribution in sector k. It is the sum of the square of firms' productivity shares in $\overline{Z_{t,k}}$: the Herfindahl index of productivities in sector k:

$$\Delta_{t,k} = \sum_{i=1}^{N_k} \left(\frac{Z_t(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_{t,k}}} \right)^2 .$$

This statistic captures the dispersion of productivity across firms in a sector. The Herfindahl index is a widely used measure of concentration. Note this Herfindahl index is among firm-level productivity and therefore is not directly observable. Because of the finite number of firms in sector k, when the productivity of firm i in sector k changes, the concentration measure $\Delta_{t,k}$ changes too: $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{2}{\Delta_k} \left(\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} - \Delta_k \right) \frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$. Note the elasticity of $\Delta_{t,k}$ with respect to $Z(k,i)$ can be positive or negative depending on the productivity level of firm i in sector k relative to the concentration measure $\Delta_{t,k}$. When the productivity of a "large" firm increases, that is for $\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} > \Delta_k$, the concentration of productivity increases. Conversely, when the productivity of a "small" firm increases, that is for $\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} < \Delta_k$, the concentration of productivity decreases.

Before describing the dynamics of these statistics under Assumption 1, I show these two statistics are sufficient to characterize the equilibrium allocation under Assumption 2.

3.2 Sector's Allocation

In this subsection, I solve for the sector-level allocation. I start by defining the sector-level markup and productivity, before characterizing the sector-level allocation under Assumption 2. Because firms' decisions are static, I abstract from the time t subscript in this section.

Markup: An important variable is the sector-level markup. This markup is defined as the sector-level price divided by the sector-level marginal cost. For a given sector k, the marginal cost is defined as $\lambda_k = \frac{dTC_k}{dY_k}$, where TC_k is the total cost in sector k: $TC_k = \sum_{i=1}^{N_k} \lambda(k,i) y(k,i)$. Note that in the context of constant return-to-scale, the marginal cost is also equal to the average cost; therefore, $\lambda_k = \frac{TC_k}{Y_k} = \sum_{i=1}^{N_k} \lambda(k,i) \frac{y(k,i)}{Y_k}$. After using the fact that firm-level price is a markup over the marginal cost, it is easy to see the sector-level markup μ_k is

$$\mu_k = \frac{P_k}{\lambda_k} = \left(\sum_{i=1}^{N_k} \mu(k,i)^{-1} s(k,i) \right)^{-1} . \tag{1}$$

The sector's markup is a sales-share-weighted harmonic average of firm-level markups. This expression is valid as long as firms charge a markup over the marginal cost. It therefore applies to any of the market structure assumed here: monopolistic, Bertrand, and Cournot.

Proposition 4 shows the sector-level markup is a function of the sector-level concentration index. In particular, the directly observable Herfindahl-Hirchman-Index (HHI), the sum of the sales share squared, plays an important role.

Proposition 4 *(**Sector-Level Markup**) The sector k's markup is equal to*

$$\mu_k = \begin{cases} \frac{\varepsilon_k}{\varepsilon_k - 1} & \textit{Under Monopolistic competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1}\left(1 - \frac{1}{\varepsilon_k - 1}\sum_{m=2}^{\infty}\left(\frac{\varepsilon_k - 1}{\varepsilon_k}\right)^{m-1}(HK_k(m))^m\right)^{-1} & \textit{Under Bertrand competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1}(1 - HHI_k)^{-1} & \textit{Under Cournot competition.} \end{cases}$$

where $HHI_k = \left(\sum_{i=1}^{N_k} s(k,i)^2\right)$ is the sector k's Herfindahl-Hirchman-Index (HHI), and $HK_k(m) = \left(\sum_{i=1}^{N_k} s(k,i)^m\right)^{1/m}$ is the Hannah and Kay (1977) concentration index. NB: $HK_k(2)^2 = HHI_k$; the HHI is the square of the second Hannah and Kay (1977) concentration index.

Proof *See Appendix A.2.* □

The above proposition shows that under monopolistic competition, the sector-level markup is constant and equal to the firm-level markup. This result is obvious because the sector's markup is an average of firms' markups, and under monopolistic competition, all the firms in a given sector charge the same markup. As soon as pricing becomes strategic, under Bertrand or Cournot competition, the sales-share distribution in the sector plays a crucial role. Under Cournot competition, for example, the HHI entirely determines the sector's markup. The intuition is as follows: when the sector's concentration is high, that is when the HHI is high, large firms have a higher market share and thus they can use this higher market power to charge higher markups, which in turn aggregate to a higher sector's markup. An important implication of Proposition 4 is that it links empirically observable variables, such as the HHI, to the sector-level markup.

Using the result in the above proposition, it is easy to derive some comparative statics of the markup with respect to the HHI while keeping everything else constant.

$$\frac{\partial \mu_k}{\partial HHI_k} = \begin{cases} 0 & \text{Under Monopolistic competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k^2}\mu_k^2 > 0 & \text{Under Bertrand competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\mu_k^2 > 0 & \text{Under Cournot competition} \end{cases}$$

Under Bertrand and Cournot competition, a higher sector's HHI always implies a higher sector's markup. This relationship is stronger for low-competitive, high-markup sectors. The sensitivity of the sector's markup to the sector's Herfindahl-Hirchman Index is stronger under Cournot than under Bertrand competition. In this framework, given the demand system and the assumed market structure, sector concentration is a measure of sector competition.

Productivity: The other important variable to define is the sector-level productivity, which is defined as the sector-level labor-augmenting productivity. As shown above, the sector-level marginal cost is $\lambda_k = \sum_{i=1}^{N_k} \lambda(k,i)\frac{y(k,i)}{Y_k}$. After substituting for the firm-level marginal cost $\lambda(k,i) =$

$Z(k,i)^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^N P_l^{\omega_{k,l}}$, the sector-level marginal cost is equal to $\lambda_k = Z_k^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^N P_l^{\omega_{k,l}}$, where

$$Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k}$$

is the sector k's (labor-augmenting) productivity, an output-weighted sum of firm-level productivity in sector k. It is entirely determined by the joint distribution of output and productivities across firms in a sector, whereas the sector-level markup is entirely determined by the distribution of sales share.

Allocation: The previous results were relating endogenous variables to each other and were not linking equilibrium allocation to the state variable in this economy. Proposition 5 first solves for the sector allocation given the sectors' markup and productivity, before explicitly describing how the two statistics, $\overline{Z_k}$ and Δ_k, entirely characterize the sector-level variables under Assumption 2.

Proposition 5 *(**Sector Allocation**) Sectors' prices are equal to:*

$$\{\log P_k\}_k = (I - \Omega)^{-1} \left\{ \log \mu_l \left(\frac{w}{Z_l} \right)^{\gamma_l} \right\}_l, \tag{2}$$

where the $(N \times N)$-matrix Ω is such that $\Omega = \{\omega_{k,l}\}_{1 \le k,l \le N}$ and I is the $(N \times N)$- identity matrix. Sectors' sales shares are equal to:

$$\left\{ \frac{P_k Y_k}{P^C C} \right\}_k' = \beta' (I - \widetilde{\Omega})^{-1}, \tag{3}$$

where the $(N \times N)$-matrix $\widetilde{\Omega}$ is such that $\widetilde{\Omega} = \{\mu_k^{-1} \omega_{k,l}\}_{1 \le k,l \le N}$ and the $(N \times 1)$- vector β is such that $\beta = \{\beta_k\}_k$. Under Assumption 2, sector k's markup and productivity are equal to:

$$\mu_k = \frac{\varepsilon_k}{\varepsilon_k - f_k(\Delta_k)} \quad \text{and} \quad Z_k = \left(\overline{Z_k}\right)^{\frac{1}{\gamma_k(\varepsilon_k - 1)}} \left(f_k(\Delta_k) \right)^{\frac{-1}{\gamma_k(\varepsilon_k - 1)}} \left(\frac{\varepsilon_k - f_k(\Delta_k)}{\varepsilon_k - 1} \right)^{\frac{-1}{\gamma_k}}$$

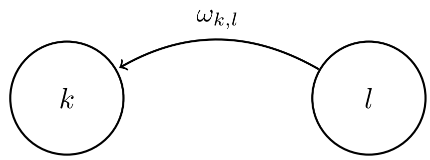
where

$$f_k(x) = \begin{cases} 1 & \text{Under Monopolistic Competition} \\ \frac{1 - \sqrt{1 - 4\left(1 - \varepsilon_k^{-1}\right)x}}{2\left(1 - \varepsilon_k^{-1}\right)x} \quad \text{for } x \in \left[0, \frac{1}{4\left(1 - \varepsilon_k^{-1}\right)}\right] & \text{Under Bertrand Competition} \\ \frac{1 - \sqrt{1 - 4(\varepsilon_k - 1)x}}{2(\varepsilon_k - 1)x} \quad \text{for } x \in \left[0, \frac{1}{4(\varepsilon_k - 1)}\right] & \text{under Cournot Competition.} \end{cases}$$

Proof *See Appendix A.3.* □

The above proposition characterizes the sectors' allocation for a given wage w. System 2 of N equations relates the sectors' prices to sectors' productivities Z_l, sectors' markups μ_l, wage w, and the I-O matrix Ω. To understand these equations, let us assume no I-O linkages exist; that is, $\Omega = 0$ and $\gamma_k = 1$. In this case, the sector k's price is just the sector's markup μ_k over the marginal cost in this sector $(w/Z_k)^{\gamma_k}$, which is standard under imperfect competition. Now let us assume the I-O structure is the one described in Figure 4; that is, sector k is using labor and sector l's good to

Figure 4: A Simple I-O Structure



NOTE: In this simple I-O structure, firms in sector k are using labor and sector l's good to produce their variety, while firms in sector l are using only labor.

produce, whereas sector l is using only labor as input. Under imperfect competition, the price in sector k is equal to the sector's markup μ_k over the marginal cost. However, sector k's marginal cost is $\left(\frac{w}{Z_k}\right)^{\gamma_k}(P_l)^{\omega_{k,l}}$, a combination of the marginal cost of labor and the price of the upstream sector l's good. Sector l's price is itself equal to the markup in sector l over the marginal cost of labor in sector l: $P_l = \mu_l\left(\frac{w}{Z_l}\right)$. To solve for the prices of sector k and l, one just needs to solve a system of two unknowns and two equations. System 2 is a generalization of this reasoning for any I-O network Ω.[14]

System 3 of N equations solves for the sectors' sales share as a function of the household expenditure share β, the markups, and the I-O network through $\widetilde{\Omega}$. To understand the intuition behind this matrix, let us assume the I-O structure is the one described in Figure 4, and let us compute the income share sector l captures from a dollar spent on sector k's good. Sector k rebates some of that dollar, a share $1-\mu_k^{-1}$, directly as profit to the household; the remaining is used to pay for inputs among which is sector l's good. Therefore, sector l receives a share $\mu_k^{-1}\omega_{k,l}$ of this dollar. For any I-O network, the element (k,l) of the matrix $\widetilde{\Omega}$ is the share of income that flows directly from sector k to sector l and is equal to $\mu_k^{-1}\omega_{k,l}$. Equation 3 shows the sales share of a sector is given by the vector $\beta'(I-\widetilde{\Omega})^{-1} = \beta' + \beta'\widetilde{\Omega} + \beta'\widetilde{\Omega}^2 + \ldots$, which captures the fact that the total sales of a sector is the sum of the direct and indirect sales to the household. A given sector receives income directly from the sales to the household. This income is captured by the term β' in Equation 3. In addition, this sector's good is also sold to its downstream sectors that used it as inputs and served the household. This first-degree indirect income is equal to the term $\beta'\widetilde{\Omega}$ in Equation 3. Furthermore, these downstream sectors' goods are also used as inputs by their own downstream sectors that sell to the household. This second-degree indirect income share is equal to the term $\beta'\widetilde{\Omega}^2$ in Equation 3. Higher-degree indirect income shares are captured in the same way by the remaining terms. The sales share of a given sector is therefore the infinite sum of these terms, which is then equal to the product of the household expenditure share β' and the Leontieff inverse of the matrix of income flow $\widetilde{\Omega}$.

Although the first part of Proposition 5 (Equations 2 and 3) does not need any specific assumption, the rest of this proposition shows that under Assumption 2, the sectors' markups and productivities are entirely determined by the two statistics, $\overline{Z_k}$ and Δ_k. Under this assumption, all the firm heterogeneity is summarized by these two statistics. Furthermore, under oligopolistic competition,

[14] For the case of perfect competition, that is, for $\mu_k = 1$, an implication of the system of equation 2 is that the sector's price is equal to the marginal cost of labor used directly and indirectly, through the I-O network.

the sector's markup is increasing in the concentration measure Δ_k. This result is very intuitive. In a given sector, when firms' productivity concentration is higher, the most productive firms have even more market power. It follows that the markup charged by these firms is even higher, which is reflected in a higher sector-level markup. An interesting result is that when the concentration measure Δ_k is converging to zero, that is, when firms become homogeneous, the markup μ_k is converging to $\varepsilon_k/(\varepsilon_k - 1)$, that is, the markup under monopolistic competition. The same is true for the sector-level productivity Z_k that converges, when Δ_k goes to zero, to $\left(\overline{Z_k}\right)^{\frac{1}{\gamma_k(\varepsilon_k-1)}}$, that is, the sector-level productivity under monopolistic competition. Therefore, the concentration measure Δ_k is capturing the intensity of competition in a sector and how much this sector market structure deviates from the Dixit and Stiglitz (1977) monopolistic competition.

Proposition 5 is important in three ways. First, this proposition solves for sector-level allocation given an equilibrium wage, nominal output, and sector-level productivities and markups. Second, it reduces firms' heterogeneity at the sector level by showing that under Assumption 2, the two statistics, $\overline{Z_k}$ and Δ_k, are sufficient to describe the sector-level allocation. Third, it gives a natural and simple interpretation to the concentration measure of productivity Δ_k, which can be though of as a measure of the competition intensity in sector k.[15]

3.3 Dynamics

In Section 3.2, the two statistics, $\overline{Z_{t,k}}$ and $\Delta_{t,k}$, have been shown to entirely described the sector-level allocation under Assumption 2. In this section, I show the dynamics of these two sufficient statistics can be summarized by a simple stochastic process under random growth at the firm-level (Assumption 1). Below, I solve for the law of motion of the firm-productivity distribution before turning to the dynamics of the two statistics, $\overline{Z_{t,k}}$ and $\Delta_{t,k}$.

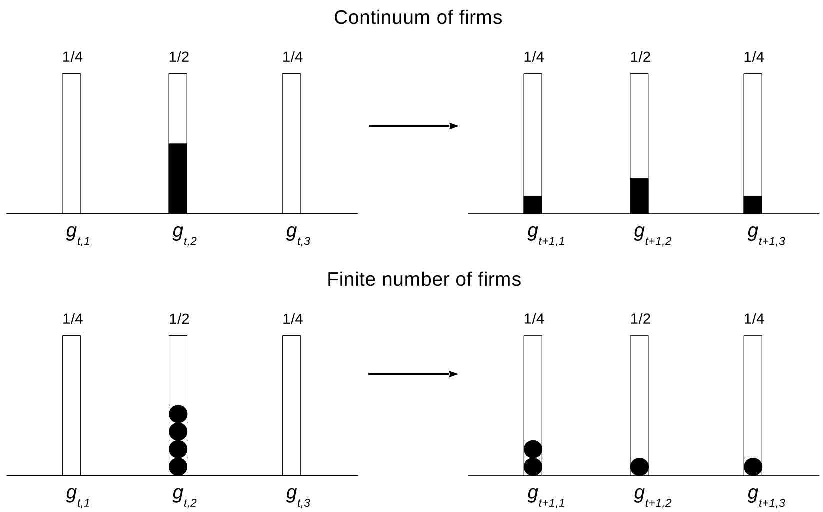
The first step is to solve for the dynamics of the distribution of productivity in each sector, the state variables of this model. Let us define the vector $g_t^{(k)} = \{g_{t,n}^{(k)}\}_{0 \leq n \leq M_k}$, where $g_{t,n}^{(k)}$ is the number of firms at productivity level φ^n at time t in sector k. The vector $g_t^{(k)}$ is thus the firm's productivity distribution at time t in sector k.[16] Recall that sector k contains an integer number of firms N_k. As in Carvalho and Grassi (2017), this assumption implies the productivity distribution is a stochastic object. To understand the intuition behind this result, let us study a simple example.

Assume only three levels of productivity and four firms exist. At time period t, these firms are distributed according to the bottom-left panel of Figure 5; that is, all four firms produce with the intermediate level of productivity. Further assume these firms have an equal probability of $1/4$ of going up or down in the productivity ladder and that the probability of staying at the same intermediate level is $1/2$. That is, the transition probabilities are given by $(1/4, 1/2, 1/4)'$. First note that, if instead of four firms, we had assumed a continuum of firms, the law of large numbers would hold such that

[15] The measure of competition intensity Δ_k is fundamentally different from other measures of competition intensity used in macroeconomics as in Aghion et al. (2014). Indeed, Δ_k is also measuring the dispersion of firm-level productivity in a sector, a fundamental of the economy.

[16] Because firm's productivity evolves on a discrete space, the vector $g_t^{(k)}$ is the histogram of the firms' productivity at time t in sector k.

Figure 5: An illustrative example of the productivity-distribution dynamics



NOTE: Top panel: with a continuum of firms, the transition is deterministic. Bottom panel: with a finite number of firms, the transition is stochastic.

at $t+1$, exactly $1/4$ of the (mass of) firms would be at the highest level of productivity, $1/2$ would remain at the intermediate level, and $1/4$ would transit to the lowest level of productivity (top panel of Figure 5). This scenario is not the case here, because the number of firms is finite. For instance, a distribution of firms such as the one presented in the bottom-right panel of Figure 5 is possible with a positive probability. Of course, many other arrangements would also be possible outcomes. Thus, in this example, the number of firms in each productivity bin at $t+1$ follows a multinomial distribution with a number of trials of four and an event-probability vector $(1/4, 1/2, 1/4)'$.

In this simple example, all firms are assumed to have the same productivity level at time t. However, extending this example to any initial arrangement of firms over productivity bins is easy. Indeed, for any initial number of firms *at a given* productivity level, the distribution of *these* firms across productivity levels next period follows a multinomial. Therefore, the *total* number of firms in each productivity level next period is simply a sum of multinomials, that is, the result of transitions from *all* initial productivity bins. The following proposition generalizes this example to determine the dynamics of the distribution of firms' productivity for any firm-level productivity process.

Proposition 6 *(**Sector k's Productivity-Distribution Dynamics**) Sector k's productivity distribution satisfies the following law of motion:*

$$g_{t+1}^{(k)} = (\mathcal{P}^{(k)})' g_t^{(k)} + \epsilon_t^{(k)}, \tag{4}$$

where $\mathcal{P}^{(k)}$ is the matrix of transition probabilities of the firm-level productivity process in sector k and where $\epsilon_t^{(k)} = \left\{\epsilon_{t,n}^{(k)}\right\}_{0 \leq n \leq M_k}$ is a mean zero random vector.

Furthermore, under Assumption 1, the stationary distribution (when $\forall t, \epsilon_t^{(k)} = 0$) is Pareto and equal to $g_n^{(k)} = N_k K_k (\varphi_k^n)^{-\delta_k}$, where K_k is a normalization constant and $\delta_k = \log \frac{a_k}{c_k} / \log \varphi_k$ is the tail index.

Proof *See Online Appendix F.2.* □

The above proposition describes the law of motion of a sector's productivity distribution and captures exactly the intuition of the example in Figure 5. The first term of the right-hand side of the law of motion 4 is the *average* behavior of the sector's productivity distribution. If an infinite number of firms were present, this *average* behavior would be exactly the next-period sector's productivity distribution. Note this term is solely a function of the current-period productivity distribution $g_t^{(k)}$ and the transition probabilities of the firm-level productivity process $\mathcal{P}^{(k)}$. Because a finite and integer number of firms are in each sector, an extra term, $\epsilon_t^{(k)}$, exists. This second term is the deviation of the actual realization of $g_{t+1}^{(k)}$ and its average behavior. As described in the example above, for a given period t, this random vector is a sum of demeaned multinomial random vectors.

A direct implication of this proposition is that sectors' productivity distribution, the state variables of this framework, are stochastic vectors. It follows that every sector's variables are stochastic and fluctuate. Finally, these sectors' productivity distributions hover around their stationary values, which are, under Assumption 1, Pareto distributed with a tail index determined by the probabilities a_k and c_k.[17]

Note that no aggregate or sector-level shocks are assumed; instead this sector- (and aggregate-) level fluctuations arise from independent firm-level shocks. The quantitative importance of such fluctuations is not formally discussed here and is addressed numerically below where the above model is calibrated to the US economy. However, the diversification among these firm-level independent shocks is weak as soon as the stationary Pareto distribution is fat-tailed. As shown by Gabaix (2011), when a small number of very productive firms exist, shocks to these firms are unlikely cancel out, and therefore they translate into quantitatively important fluctuations.[18]

After the characterization of the dynamics of the sector's productivity distribution (Proposition 6), the second step is to describe the law of motion of the two statistics, $\overline{Z_{t,k}}$ and $\Delta_{t,k}$, that are sufficient under Assumption 2. Proposition 7 below shows that under random growth (Assumption 1), the law of motion of these two statistics can be described by a simple process.

Proposition 7 ***(Dynamics of $\overline{Z_{t,k}}$ and $\Delta_{t,k}$)*** *Under Assumption 1, the two statistics, $\overline{Z_{t,k}}$ and $\Delta_{t,k}$, of the sector k's productivity distribution satisfy the following dynamics:*

$$\overline{Z_{t+1,k}} = \rho_k^{(Z)}\overline{Z_{t,k}} + o_{t,k}^{(Z)} + \sqrt{\varrho_k^{(Z)}\Delta_{t,k} + O_{t,k}^{(Z)}} \quad \overline{Z_{t,k}} \quad \varepsilon_{t+1,k}^{(Z)}$$

$$\left(\frac{\overline{Z_{t+1,k}}}{\overline{Z_{t,k}}}\right)^2 \Delta_{t+1,k} = \rho_k^{(\Delta)}\Delta_{t,k} + o_{t,k}^{(\Delta)} + \sqrt{\varrho_k^{(\Delta)}\chi_{t,k} + O_{t,k}^{(\Delta)}} \quad \Delta_{t,k} \quad \varepsilon_{t+1,k}^{(\Delta)}$$

where $\varepsilon_{t+1,k}^{(Z)}$ and $\varepsilon_{t+1,k}^{(\Delta)}$ are random variables following a $\mathcal{N}(0,1)$ with a non-zero covariance and where $\chi_{t,k}$, $o_{t,k}^{(Z)}$, $o_{t,k}^{(\Delta)}$, $O_{t,k}^{(Z)}$, and $O_{t,k}^{(\Delta)}$ are predetermined at time t, whereas, $\rho_k^{(Z)}$, $\rho_k^{(\Delta)}$, $\varrho_k^{(Z)}$, and $\varrho_k^{(\Delta)}$ are constant.

[17] The concept of a stationary distribution is the same as in Hopenhayn (1992) and Hopenhayn and Prescott (1992).

[18] In a one-sector model with perfect competition and entry/exit à la Hopenhayn (1992), Carvalho and Grassi (2017) study the behavior for an increasingly large number of firms of the volatility arising from idiosyncratic independent shocks on firms.

Proof *See Online Appendix F.3.* □

Proposition 7 is similar to Theorem 2 of Carvalho and Grassi (2017). It shows the dynamics of the two moments of sector k's productivity distribution are persistent. The intuition is that because the firm-level productivity is persistent, persistence is aggregated at the sector level. The higher the firm-level persistence, the higher the sector-level persistence, as shown in Carvalho and Grassi (2017).[19]

Moreover, the (conditional) variance of the sum of the productivity of sector k's firms, $\overline{Z_{t,k}}$, is time varying and is determined by the Hearfindahl index of firms' productivity $\Delta_{t,k}$. Here, as in Gabaix (2011) and Carvalho and Grassi (2017), any volatility at the sector level is due to idiosyncratic shocks at the firm level. When a sector is concentrated, shocks to firms with a large productivity do not wash out at the aggregate level. A higher concentration implies a higher importance of these firms and thus more volatility due to idiosyncratic shocks.

4 Structural Firms

In this section, I show the structural importance of a firm is determined by the firm's size, the firm's sector industrial organization, and its role in the I-O network. The structural importance of a firm is defined here as the elasticity of aggregate output with respect to the productivity of one firm in one sector.

To compute this elasticity, the first step is to solve for the aggregate output and the equilibrium wage. I assume the household supplies inelastically one unit of labor, and I normalized the price of the composite consumption good to 1. The following proposition describes the equilibrium allocation given sector-level markups and productivities.

Proposition 8 *(**Equilibrium Allocation**) For given sector-level markups μ_k and productivities Z_k, the wage is*

$$\log w = -\beta'(I-\Omega)^{-1}\left\{\log \mu_k Z_k^{-\gamma_k}\right\}_k = -\sum_{k=1}^{N}\overline{\beta_k}\log \mu_k Z_k^{-\gamma_k}, \tag{5}$$

where β is the $(N\times 1)$-vector of the household expenditure share $\{\beta_k\}_k$ and $\left\{\overline{\beta_k}\right\}_k' = \beta'(I-\Omega)^{-1}$ are the sectors' centrality. The share of aggregate profit in nominal output is

$$\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\left\{1-\mu_k^{-1}\right\}_k = \sum_{k=1}^{N}\beta_k\left(1-\widetilde{\mu_k}^{-1}\right), \tag{6}$$

where $\widetilde{\Omega} = diag\left(\left\{\mu_k^{-1}\right\}_k\right)\Omega$ with $diag\left(\{x_k\}_k\right)$ is the diagonal matrix whose non-zero elements are the x_k and where $\widetilde{\mu_k}$ is such that $\left\{1-\widetilde{\mu_k}^{-1}\right\}_k = (I-\widetilde{\Omega})^{-1}\left\{1-\mu_l^{-1}\right\}_l$. Finally, aggregate output is

$$\log Y = \log w - \log\left(1-\frac{Pro}{P^C C}\right). \tag{7}$$

[19] For $n\in\mathbb{N}^*$, the sequences $v_{k,n} = a_k\varphi_k^{-n(\varepsilon_k-1)\gamma_k} + b_k + c_k\varphi_k^{n(\varepsilon_k-1)\gamma_k}$ and $w_{k,n} = a_k\varphi_k^{-2n(\varepsilon_k-1)\gamma_k} + b_k + c_k\varphi_k^{2n(\varepsilon_k-1)\gamma_k} - (\rho_k^{(n)})^2$ are, respectively, the mean and variance of the growth rate of firm i in sector k productivity measure $Z(k,i)^{n(\varepsilon_k-1)\gamma_k}$. We have that, $\rho_k^{(Z)} = v_{k,1}$, $\rho_k^{(\Delta)} = v_{k,2}$, $\varrho_k^{(Z)} = w_{k,1}$, and $\varrho_k^{(\Delta)} = w_{k,2}$.

Proof *See Appendix A.4.* □

The equilibrium wage of Equation 5 comes from sectors' price (Equations 2) and the normalization $P^C = 1$. Note that the log of wage can be rewritten as a weighted sum of sector-level markups and productivities where the weights are $\left\{\overline{\beta_k}\right\}_k' = \beta'(I-\Omega)^{-1} = \beta'(I+\Omega+\Omega^2+\ldots)$ the sectors' *centrality*. The centrality measures the direct and indirect importance of a sector in the household consumption bundle. A sector's good contributes to the consumption bundle by the direct consumption of this good by the household. This direct contribution is governed by the shares β. This good is also used as input by other sectors that are consumed by the household. This first-degree indirect contribution to the household consumption bundle is captured by the term $\beta'\Omega$. Furthermore, this good is also used as an input for other goods that are inputs of other goods that are consumed by the household. This second-degree indirect importance is captured by the term $\beta'\Omega^2$. Higher-degree linkages are captured in the same way. The centrality is then the infinite sum of these terms, which is then equal to the product of the share β and the Leontieff inverse $(I-\Omega)^{-1}$. The centralities $\overline{\beta}_k$ take into account the direct and indirect consumption of a sector's good through the I-O network.

The aggregate profit share is a function of sectoral markups and the I-O network. To understand the intuition behind Equation 6, let us compute the profit share of one dollar spent on sector k in the simple I-O network of Figure 4. The sector-level markup determines the profit share: a share $1-\mu_k^{-1}$ of this dollar is directly rebated to the household as profit. The remaining, μ_k^{-1}, is used to pay for inputs, among which is the sector l's good. Therefore, sector l receives $\mu_k^{-1}\omega_{kl}$ of income of every dollar spent on sector k, from which a share $1-\mu_l^{-1}$ is rebated to the household. The total profit rebated to the household of this dollar spend on sector k is then equal to $1-\mu_k^{-1}+\mu_k^{-1}\omega_{kl}\left(1-\mu_l^{-1}\right)$. Equation 6 is a generalization of this intuition to any I-O structure. The element $\mu_k^{-1}\omega_{k,l}$ of the matrix $\widetilde{\Omega}$ is the income share that goes from sector k to sector l. The Leontieff inverse of this matrix, $(I-\widetilde{\Omega})^{-1}$, gives the direct and indirect income share that goes from one sector to another whereas the vector $\left\{1-\mu_k{}^{-1}\right\}_k$ gives the income share of each sector that is directly rebated to the household. The aggregate profit share can also be rewritten as a weighted sum of the expenditure share β_k, where the weights $1-\widetilde{\mu_k}^{-1}$ are the direct and indirect profit share of each sector k. Note also that $\widetilde{\mu_k}^{-1}$ is the direct and indirect labor share of each sector, and it is such that $\{\widetilde{\mu_k}\}_k = (I-\widetilde{\Omega})^{-1}\{\gamma_k\mu_k^{-1}\}_k$.[20]

The aggregate output equation comes from the household budget constraint and the inelastic labor supply. Note this equation will be different for different utility functions. Appendix D derives the case of elastic labor supply for both separable and Greenwood–Hercowitz–Huffman (GHH) preferences. Under Assumption 2, the results in Propositions 8 and 5 describe entirely the equilibrium allocation as a function of the two sufficient statistics, $\overline{Z_k}$ and Δ_k. The first part of Proposition 5 and Proposition 8 solve for the equilibrium allocation as a function of sector-level markups and productivities, whereas the second part of Proposition 5 gives the sectors' markup and productivity as a function of these two sufficient statistics.

Let us decompose the effect of an increase in productivity of one firm in one sector on aggregate output into the "downstream" and "upstream" parts of aggregate output. The "downstream" part of

[20] This can be shown using the definition of $\widetilde{\mu_k}^{-1}$ and the fact that $\widetilde{\Omega}\{1\}_k = diag(\{\mu_k^{-1}\}_k)\Omega\{1\}_k = \{\mu_k^{-1}(1-\gamma_k)\}_k$.

aggregate output is defined as the first term of the right-hand side of Equation 7: $\log Y^d = \log \frac{w}{P}L = \log w$. This term is the (log) real labor income, because total labor and the composite good price are normalized to 1. The "upstream" part is defined as $\log Y^u = -\log\left(1 - \frac{Pro}{P^C C}\right) = -\log\left(\frac{wL}{P^C C}\right)$, which is the (log of) the labor share. Therefore, this decomposition of aggregate output is just a decomposition in terms of the real labor income and the labor share. The terminology "downstream" comes from the fact that any change in a sector's price impacts the downstream sectors and is reflected in the wage. The "upstream" terms comes from the fact any change in markups and thus cost share impact the income share received by the upstream sectors and is ultimately reflected in the aggregate profit/labor share.[21] The elasticity of aggregate output to the productivity $Z(k, i)$ of firm i in sector k is then the sum of the effect on the "downstream" and "upstream" parts of aggregate output:

$$\underbrace{\frac{\partial \log Y}{\partial \log Z(k,i)}}_{\text{change in GDP}} = \underbrace{\frac{\partial \log Y^d}{\partial \log Z(k,i)}}_{\text{change in labor income}} + \underbrace{\frac{\partial \log Y^u}{\partial \log Z(k,i)}}_{\text{-change in labor share}}$$

First, let us look at the effect of a change in productivity of one firm in one sector on the labor income. The change in the "downstream" part of aggregate output captures any change in the real wage. These changes are themselves due to changes in sectoral prices. Changes in sectoral prices propagate to downstream sectors. To understand the intuition, let us once again look at the simple I-O of Figure 4. Recall that in this simple case, sector k's price is

$$\log P_k = \log \mu_k + \log\left(\frac{w}{Z_k}\right)^{\gamma_k} + \omega_{k,l} \log P_l \quad \text{with} \quad \log P_l = \log \mu_l + \log\left(\frac{w}{Z_l}\right).$$

Following a change in the productivity of one firm in sector l, the two statistics, $\overline{Z_l}$ and Δ_l, are affected as described in Section 3.1. These changes affect the markup μ_l and the productivity Z_l in sector l (Proposition 5), which in turn affect the price in sector l. Any change in sector l's price impacts the marginal cost of the downstream sector k and therefore the price of sector k. Any shocks to firms in a sector propagate to downstream sectors through the price. This shocks ultimately affects price in all downstream sectors and thus the real wage that is, the "downstream" part of aggregate output. The strength of this effect depends on (i) the pass-through in sector l, that is, on how much sector l's price changes after the increase in productivity of one of its firms, and on (ii) the I-O linkage between sector l and sector k. The market structure of the sector and the identity of the firms whose productivity increases determine the strength of the pass-through. Proposition 9 computes the elasticity of the "downstream" part of aggregate output with respect to the productivity of one

[21] See Section 5 for a study of the propagation of firm-level shocks across sectors.

firm in one sector for any I-O network.

Proposition 9 *(**Elasticity "downstream"**) Assume 2, the elasticity of the "downstream" part of aggregate output with respect to the productivity of firm i in sector k is*

$$\frac{\partial \log Y^d}{\partial \log Z(k,i)} = \frac{\overline{\beta_k}}{\varepsilon_k - 1}\left(1 + \frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)},$$

where $e_k = \frac{d\log f_k}{d\log \Delta_k}$ and $\left\{\overline{\beta_k}\right\}'_k = \beta'(I-\Omega)^{-1}$ is the vector of sectors' centrality.

Proof *See Appendix A.5.* □

The elasticity of the "downstream" part of aggregate output with respect to the productivity of firm i in sector k is the product of three terms. The first term is the sector's centrality $\overline{\beta_k}$. As discussed earlier, the centrality measures the direct and indirect importance of a sector in the household consumption bundle. The second term captures the effect of oligopolistic competition; under monopolistic competition this term would be equal to 1. Whenever firm i in sector k is "large", that is, when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} > \Delta_k$, this second term is smaller than 1, because, when the productivity of a large firm increases, some of the productivity gains translate into an increase in the markup rather than a decrease in price. Indeed, this firm already has a lot of market power and does not need to cut its price and increase its production by as much as under monopolistic competition: the pass-through is incomplete. At the sector level, the price falls by less than under monopolistic competition and the effect on the "downstream" part of aggregate output is smaller. Conversely, if the productivity of a "small" firm increases, that is, when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} < \Delta_k$, the second term is larger than 1. When the productivity of this firm increases, it decreases its price and increases its markup but also cuts the markups of larger firms. At the sector level, the price falls by more than under monopolistic competition and the effect on the equilibrium wage and the "downstream" part of aggregate output is stronger. The last term is the effect of the firm's increase in productivity on the first moment of the sector k productivity distribution. The more productive the firm affected, the larger this term.

One can see that the elasticity of the "downstream" part of aggregate output reflects the effect of the change in price on the real wage following an increase in $Z(k,i)$ by rewriting it as[22]:

$$\frac{\partial \log Y^d}{\partial \log Z(k,i)} = \overline{\beta_k}\left(\frac{1}{\varepsilon_k - 1}\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} - \frac{\frac{\varepsilon_k}{\varepsilon_k-1} - 1}{\mu_k - 1}\frac{\partial \log \mu_k}{\partial \log Z(k,i)}\right) \tag{8}$$

The centrality $\overline{\beta_k}$ is the importance of the firm's sector good in the determination of the wage. The first term in the bracket is equal to the change in the sector k's price under monopolistic competition following the increase in the productivity of firm i. The second term in the bracket is the change in sector's markup due to the change in the firm i's productivity. This last term would be equal to zero under perfect competition. As described earlier, this term can be either negative or positive depending on the identity of the firm subject to the shock.

[22] Note that using the expression of the markup under Assumption 2 (Proposition 5) and the elasticity of Δ_k to $Z(k,i)$ in Section 3.1, we can show that the elasticity of the markup is $\frac{\partial \log \mu_k}{\partial \log Z(k,i)} = -(\mu_k - 1)\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)$.

The structural importance of a firm for the "downstream" part of aggregate output is a function of the I-O network through the sector's centrality, the sector's market structure index by the Δ_k, and, the firm size through the term $\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$. The market structure and the firm size govern the change in the sector's price following a change in the firm's productivity, whereas the I-O linkages determine the intensity of the effect of this change in the sector's price on other sectors and on the equilibrium wage.

Let us now study the effect of a change in productivity of one firm in one sector on the labor share. The change in the "upstream" part of aggregate output captures any change in the aggregate labor and profit income share. These changes are themselves due to changes in sectoral profit share. Changes in the sectoral profit share affect the income received by the upstream sectors. To understand this effect, let us look at the profit share of a dollar spent on sector k's good in the simple I-O structure of Figure 4. In this example, the share of profit of one dollar spent on sector k's good is $1 - {\mu_k}^{-1} + {\mu_k}^{-1}\omega_{kl}\left(1 - {\mu_l}^{-1}\right)$. Following a shock to the productivity of firm i in sector k, the statistic Δ_k changes, and let assume it increases. This increase in Δ_k increases μ_k, the markup in sector k (Proposition 5). As a consequence, less income goes toward paying for inputs, among which is sector l's good. The total share of profit/labor is affected because (i) the sector k rebates more profit to the household and (ii) the upstream sector l receives less income and therefore rebates less profit to the household. Proposition 10 generalizes the above intuition to any I-O structure.

Proposition 10 *(**Elasticity "upstream"**) Assume 2, the elasticity of the "upstream" part of aggregate output with respect to the productivity of firm i in sector k is*

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = -\frac{P^C C}{wL}\frac{P_k Y_k}{P^C C}\frac{(\mu_k - 1)}{\widetilde{\mu_k}}\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}.$$

where $e_k = \frac{d \log f_k}{d \log \Delta_k}$ is the elasticity of f_k, and where $\widetilde{\mu_k}$ is such that $\left\{1 - \widetilde{\mu_k}^{-1}\right\}_k = (I - \widetilde{\Omega})^{-1}\left\{1 - \mu_l^{-1}\right\}_l$ with $\widetilde{\Omega} = diag\left(\left\{\mu_k^{-1}\right\}_k\right)\Omega$.

Proof *See Appendix A.5.* □

The elasticity of the "upstream" part of aggregate output is the product of several terms. The first important term, $\widetilde{\mu_k}^{-1}$, is the cost share of sector k's income that is rebated as labor income to the household directly and indirectly through other sectors. The second important term is proportional to the change in sector's cost/profit share, $\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$. Under monopolistic competition, that is, when $\Delta_k \to 0$, this term is zero. Under oligopolistic competition, this term can be either positive or negative. Whenever firm i in sector k is "large", that is, when $\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} > \Delta_k$, this term is negative and $\frac{\partial \log Y^u}{\partial \log Z(k,i)}$ becomes positive. This result is very intuitive: when the productivity of a large firm increases, this firm reduces its price but also uses its market power to raise its markup. At the sector level, the profit share is higher, which translates to a higher (resp. lower) aggregate profit share (resp. labor share) and a higher "upstream" part of aggregate output. Conversely, when the productivity of a "small" firm increases, that is, when $\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} < \Delta_k$, this term is positive and the elasticity of the "upstream" part of aggregate output is negative, because when the productivity of a "small" firm increases, this firm decreases its price and increases its markup but also cuts the markup of larger firms. At the sector level, the markup is reduced, which translates into smaller (resp. larger) profit share (resp. labor share). The aggregate profit share (resp. labor share) is therefore reduced, as is the "upstream" part of aggregate output. The elasticity of the "upstream" part of aggregate output reflects the effect of the change in cost share on the aggregate labor share following an increase in $Z(k,i)$. One can rewrite this elasticity as follows:

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = - \quad \frac{P_k Y_k \widetilde{\mu_k}^{-1}}{wL} \quad \frac{\partial \log(\mu_k^{-1})}{\partial \log Z(k,i)}.$$

This expression shows the effect on the aggregate labor income, the upstream part of aggregate output, is determined by sector k's direct and indirect labor share and the elasticity of sector k's cost share. Sector k's direct and indirect labor share is measured as a share of the total labor income by the sales of sector k, $P_k Y_k$, of which a share $\widetilde{\mu_k}^{-1}$ is rebated directly and indirectly as labor income. Sector k's cost share is μ_k^{-1}. Because of oligopolistic competition, this share is affected by changes in the productivity of firms in sector k.

Both the market structure and the I-O network impact the propagation of firm-level shocks on the "upstream" part of aggregate output. The markup centrality $\widetilde{\mu_k}$ is jointly determined by the I-O network and the competition intensity through the matrix $\widetilde{\Omega}$, whose elements gives the income share that flows between two sectors.

In conclusion, the structural importance of a firm is determined by the firm size, the market structure, and the I-O network. The firm size determines the firm's influence on the sector's price and profit share. The I-O network determines the sector's importance in the consumption bundle and the aggregate profit/labor share. The sector's market structure interacts with both the firm size and the I-O network in shaping the firm's structural importance. Indeed, with the firm size, it governs the strength of the change in the sector's price and profit share following a shock to one firm, and, with the I-O network, it governs a sector's importance for the aggregate profit/labor share.

The decomposition between the "downstream" and "upstream" parts of aggregate output is valid because I assume an inelastic labor supply. In this case, any change in profit/labor share does not feed back to the wage by affecting the labor supply, whereas the effect of a wage increase on aggregate output is not magnified by an endogenous increase in labor supply. Relaxing the inelastic-labor-supply assumption won't affect the results, but it affects the interpretation of the terms "downstream" and "upstream". As Appendix D shows, with separable or GHH preferences, the output is still a function of the wage and the profit/labor share. They are themselves only functions of sectoral productivities and markups (Proposition 8) and therefore, under Assumption 2, a function of the statistics $\overline{Z_k}$ and Δ_k (Proposition 5).

5 Propagation

In this section, I show how a shock to one firm in one sector propagates to other sectors through the I-O network. The propagation to downstream sectors is due to changes in price whose magnitude is governed by the competition intensity. The new propagation mechanism to upstream sectors is entirely due to the endogenous changes in cost/profit share. To study the propagation of firm-level productivity shocks in the economy, I derive the elasticity of sector-level price (Proposition 11) and sales share (Proposition 12) with respect to the productivity of one firm in one sector. These results, together with the elasticity of aggregate output derived in the previous section (Propositions 9 and 10), allow the derivation of the effect of an increase in productivity of one firm in a sector on sector-level output.

The effect of a change in productivity of one firm on other sectors' price is summarized in Proposition 11 by the elasticity of sector k's price with respect to the productivity of firm j in sector l.

Proposition 11 *(**"Downstream" Propagation**) Assume 2, the elasticity of the sector k's price with respect to the productivity of firm j in sector l is*

$$\frac{\partial \log P_k}{\partial \log Z(l,j)} = \frac{\partial \log w}{\partial \log Z(l,j)} - \frac{\psi^d_{k,l}}{\varepsilon_l - 1}\left(1 + \frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\right)\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)}$$

where $e_k = \frac{d\log f_k}{d\log \Delta_k}$ is the elasticity of f_k, and $\psi^d_{k,l}$ is the element (k,l) of the matrix $\psi^d = (I-\Omega)^{-1}$.

Proof *See Appendix A.6.* □

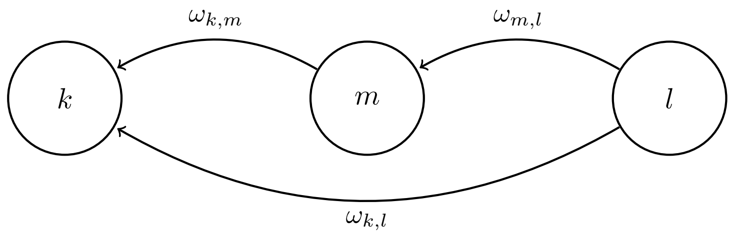
The change in sector k's price reflects the change in cost. It is the sum of the change in the cost of labor and intermediate goods. Following a change in the productivity of firm j in sector l, the wage changes as it is described in the previous section and in Proposition 9. This change in wage is the structural importance of the firm j in sector l on the wage. This change in wage affects the cost of labor and thus sector k's price.[23]

The change in the intermediate-goods price is captured by the second term, which is the product of (i) the direct and indirect exposure of sector k's production to sector l's good, (ii) the effect of oligopolistic competition in sector l, and (iii) the effect of the change in productivity of firm j on sector l's productivity. The degree of direct and indirect exposure of sector k's production to sector l's good is measured by $\psi^d_{k,l}$, the element k,l of the matrix $(I-\Omega)^{-1}$. In the simple I-O structure of Figure 4, this parameter is exactly equal to $\omega_{k,l}$. For a more general I-O structure, the number $\psi^d_{k,l}$ captures the dependence of sector k's production on sector l's good directly and through other sectors. For example, let us assume the I-O network is as in Figure 6; that is, sector k is using sector l and sector m's goods to produce, and sector m is also using sector l's good to produce. In this simple case, the direct and indirect exposure of sector k's production to sector l's good, $\psi^d_{k,l}$, takes into account the direct consumption of sector l's good by sector k plus the indirect consumption of sector l's good through sectors m, because the latter is also using l to produce its good: $\psi^d_{k,l} = \omega_{k,l} + \omega_{k,m}\omega_{m,l}$.

The effect of oligopolistic competition is similar to the one described in the previous section. It measures the pass-through of the increase in productivity of firm j on sector l's price. Under monopolistic competition, the term $1 + \frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)$ would be equal to 1. Here, depending on

[23] Note this change in wage affects all the sectors in the economy and therefore also indirectly affects sector k through its intermediate input consumption, which is the reason why no parameter γ_k, the labor share in sector k, is in front of the term $\frac{\partial \log w}{\partial \log Z(l,j)}$.

Figure 6: An Example of Input-Output Structure



NOTE: In this simple I-O structure, firms in sector k are using labor, and sector l and sector m's goods to produce their variety, firms in sector m are using labor and sector l's good, and firms in sector l are using only labor.

the identity of the firm j, the response of sector l is larger or smaller than under monopolistic competition. If the productivity of a large firm increases, that is, $\frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}} > \Delta_l$, some of the increase in productivity translates into an increase in markup at the sector level, and therefore sector l's price falls by less than under monopolistic competition. Conversely, if the increase in productivity affects a small firm, that is, $\frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}} < \Delta_l$, this firm is cutting the markup of its larger competitors and thus reduces the markup at the sector level. In this case, sector l's price falls by more than under monopolistic competition. To understand this result more clearly, let us rewrite the elasticity of sector k's price as:

$$\frac{\partial \log P_k}{\partial \log Z(l,j)} = \frac{\partial \log w}{\partial \log Z(l,j)} - \psi^d_{k,l} \left(\frac{1}{\varepsilon_l - 1} \frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)} - \frac{\frac{\varepsilon_l}{\varepsilon_l - 1} - 1}{\mu_l - 1} \frac{\partial \log \mu_l}{\partial \log Z(l,j)} \right).$$

The effect on sector k's price is larger or smaller depending on the sign of the elasticity of sector l's markup with respect to $Z(l,j)$. As described above, the sign of this elasticity is a function of the identity of firm j.

Without an I-O network, the term $\psi^d_{k,l}$ would be replaced by one for $k = l$ and by zero otherwise that is, a shock to one firm in a sector affects other sectors only through the effect on wage. With an I-O network but without oligopolistic competition, sector l's markup would be constant, that is, $\frac{\partial \log \mu_l}{\partial \log Z(l,j)} = 0$. In that case, only the size of the firm would matter through $\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)}$, and the market structure in the sector would be irrelevant.

Let us now look at the effect of a change in productivity of one firm on other sectors' sales share. This effect is summarized in Proposition 12 by the elasticity of sector k's sales share with respect to the productivity of firm j in sector l.

Proposition 12 ***("Upstream" Propagation)*** *Assume 2, the elasticity of the sector k's sales share with respect to the productivity of firm i in sector l is*

$$\frac{\partial \log\left(\frac{P_kY_k}{P^CC}\right)}{\partial \log Z(l,j)} = (\psi^s_{l,k} - \mathbb{I}_{l,k})\frac{P_lY_l}{P_kY_k}(\mu_l - 1)\frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\frac{\partial \log \overline{Z}_l}{\partial \log Z(l,j)},$$

where $e_k = \frac{d\log f_k}{d\log \Delta_k}$ is the elasticity of f_k, $\mathbb{I}_{l,k}$ is equal to 1 if $k = l$ and zero otherwise ,and $\psi^s_{l,k}$ is the element (l,k) of the matrix $\psi^s = (I - \widetilde{\Omega})^{-1}$.

Proof *See Appendix A.7.* $\square$

The change in sector k's sales share reflects the change in demand from sector l. The demand from sector l is determined by the total cost share of sector l and the exposure of sector k to sector l demand. Any change in the cost share is a change in the opposite sign of the profit share. For example, after an increase in the profit share, more income is rebated to the household as profit and less income is used to pay for inputs. Following an increase in productivity of firm j in sector l, the profit share changes, and depending on the identity of firm j, it can increase or decrease. If j is large, that is, $\Delta_l < \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}$, the gain in productivity translates into an increase in markup and thus of the profit share. Conversely, if j is small, that is, $\Delta_l > \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}$, the increase in firm j's productivity cuts the markup of its larger competitors in sector l, and the profit share at the sector level increases. Sector l's cost share is μ_l^{-1}, and rewriting the elasticity of the sector k's sales share as a function of the change in cost share folows:

$$\frac{\partial \log\left(\frac{P_kY_k}{P^CC}\right)}{\partial \log Z(l,j)} = (\psi^s_{l,k} - \mathbb{I}_{l,k})\frac{P_lY_l}{P_kY_k}\frac{\partial \log \mu_l^{-1}}{\partial \log Z(l,j)}. \tag{9}$$

This elasticity is the product of three terms. The last two terms represent the change in total cost in sector l as a share of sector k's sales share. It is the product of sector l's income and the change in sector l's cost share following an increase in productivity of firm j in sector l. The first term represents the exposure of sector k to the direct and indirect demand of sector l. The number $\psi^s_{l,k}$, that is, the element (l,k) of the matrix $(I - \widetilde{\Omega})^{-1}$, is the share of sector l's income that goes to sector k directly or indirectly through other sectors. To understand this results, let us assume the I-O network is the simple one of Figure 6. In this case, the number $\psi^s_{k,l}$ takes into account the direct income share used by sector k to pay for the input of sector l's good. It also takes into account the indirect income share of sector k that goes to sector l through sector m because the latter is also using sector l's good to produce. In the simple case of Figure 6, we have $\psi^s_{k,l} = \mu_k^{-1}\omega_{k,l} + \mu_k^{-1}\omega_{k,m}\mu_m^{-1}\omega_{m,l}$.

Note sector k's sales share would be kept constant if no I-O network exists. In that case, $\psi^s = I$ and the first term of Equation 9 would be zero. Without an I-O network, all the demand comes from the households that are spending a constant share of their income on each good, thanks to the Cobb-Douglas preferences. Even if the increase in productivity of firm j in sector l is affecting aggregate income (see Section 4), it is not affecting one sector more than the others. Note also that sector

k's sales share would be constant if monopolistic competition were assumed. In that case, sector l's cost share would be fixed and the last term of Equation 9 would be zero. Without oligopolistic, competition the change in productivity of firm j in sector l does not affect the sector-level markup, and the repartition of income between inputs and profit is fixed by the value of the parameter ε_k. The propagation mechanism of Proposition 12 requires both an I-O network and an endogenous market structure to operate.

Proposition 11 relates to the "downstream" propagation because sector l's shock affects sector k strongly for higher values of $\psi^d_{k,l}$. This term is the element (k, l) of the matrix $(I - \Omega)^{-1}$, which measures the direct and indirect cost share of sector l's good in sector k production: the good is going from sector l to sector k; that is, sector k is downstream from sector l. Proposition 12 relates to the "upstream" propagation because a shock to sector l affects sector k strongly for higher values of $\psi^s_{l,k}$. This term is the element (l, k) of the matrix $(I - \widetilde{\Omega})^{-1}$, which measures the direct and indirect income share of sector k's good in sector l production: the good is going from sector k to sector l; that is, sector k is upstream from sector l. Note that here this notion of "downstream" and "upstream" are a generalization of the usual definition to take into account the indirect linkages between sectors.

6 Quantitative Results

In this economy, all aggregate uncertainty comes from the firm-level productivity stochastic process $Z_t(k, i)$. Because the number of firms in a sector is finite, firm-level fluctuations translate into sector-level fluctuations. However, these firms play an oligopolistic competition game and take into account the impact of their decisions on their sector. This results in incomplete pass-through of shocks to price and fluctuations in profit share. When the firm-level productivity process is assumed to follow random growth (Assumption 1), the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$ are stochastic and follow $AR(1)$-type processes (Proposition 7). While under Assumption 2 and under oligopolistic competition, fluctuations in these statistics create fluctuations in sector-level markups and productivities according to Proposition 5. The origin of these sector-level fluctuations is "granular" (Gabaix, 2011) and they are due to the presence of large firms in a given sector. Sectors are linked through a "small world" I-O network (Figure 1) that contains a handful of hub-like sectors. Similar to Acemoglu et al. (2012) and Carvalho (2010), sector-level fluctuations do not average out, and create sizable fluctuations in output as computed in Proposition 8.

In this section, I evaluate the quantitative importance of firm-level productivity shocks and oligopolistic competition in shaping the business cycle. To this end, I first calibrate the above framework to the US economy. I then simulate a path of firm-level productivity for each firm, and I solve for the equilibrium allocation in each period. I then compute business-cycle statics and decompose variance of aggregate output into fluctuations in labor income and labor share, that is, into the "downstream" and "upstream" part of aggregate output. Finally, I decompose the variance of sector-level output into fluctuations of price and sales share.

6.1 Calibration and Numerical Strategy

To calibrate this economy, the first step is to choose preferences and deep parameters of the model. Consistent with the analysis in Section 4, I assume labor supply is inelastic. Such an assumption allows me to interpret the decomposition of aggregate output between the labor income and labor-income share as "downstream" and "upstream".[24] Appendix D shows how relaxing the inelastic-labor-supply assumption affects the results. The second important assumption is the choice of the parameter ε_k. I choose this parameter to be equal to 5 in every sector: $\forall k, \varepsilon_k = 5$. Even if this assumption is strong, this value seems reasonable because the international trade literature has estimated this number to be between 3 and 9 (see Imbs and Mejean (2015) for a review). Note the estimates in the international trade literature are not necessarily consistent with the above model, because they are usually not assuming oligopolistic competition. An elasticity of 5 across varieties within a sector implies a sectoral markup for monopolistic competition of 1.25. Finally, I assume the sector competition to be differentiated Bertrand.

The second step consists of using concentration and I-O data to discipline the sector-level parameters. For this calibration, a sector is an industry as defined by the BEA in its detailed I-O classification. The BEA identifies 389 sectors and the level of disagregation is comparable for most sectors to the 5-digit NAICS classification. This dataset is the most disaggregated level available with sectoral I-O linkages information. The US Census Bureau gives information on sector-level concentration. I use the 2007 vintage of these data and especially the HHI of sales share among the top 50 firms, namely, HHI_k.[25] Under Assumption 2 and using Proposition 4, we can easily show $\left(1 - \varepsilon_k^{-1}\right) HHI_k = f_k(\Delta_k) - 1$.[26] I use this relationship and the value of ε_k to back out the concentration measure of productivity at the sector level Δ_k. With these values in hand, I calibrate the Markov chain of the productivity process $\mathcal{P}^{(k)}$ to match a firm-level volatility of $\sigma_k = 0.1$ and the value of Δ_k. The firm-level volatility is at the lower hand of the estimate in the firm dynamics literature.[27] Note, however, that the literature's estimates are not necessarily consistent with the oligopolistic competition assumption in the framework presented in this paper. The matrix Ω is calibrated using the latest vintage of the detailed I-O data of the BEA for 2007. For a description of the data, see Appendix B. The model counterpart of the I-O table provided by the BEA is $\widetilde{\Omega}$, whose elements are the share of *income* that goes from one sector to the other. Using the concentration, I compute the sector-level markup, and I use the relation between Ω and $\widetilde{\Omega} = diag(\{\mu_k^{-1}\}_k)\Omega$ to recover the actual Ω whose elements are the share of *total cost* that goes from one sector to the other. As Figure 1 shows and as Acemoglu et al. (2012) or Carvalho (2014) show, this I-O network is a "small-world" network in which a handful of sectors are heavily connected to the other sectors.

Table 2 summarizes the parameters of the baseline calibration. This calibration has $N = 389$ sectors.

[24] This assumption is also standard in the literature on the micro-origin of aggregate fluctuations, and allow, for comparison with Acemoglu et al. (2012), Baqaee (2016), or Baqaee and Farhi (2017a). Carvalho and Grassi (2017) assume elastic labor supply, and their results should be compared with the results in Appendix D.

[25] These data are described in Appendix B.

[26] Note that at the second order, we have $\left(1 - \varepsilon_k^{-1}\right) HHI_k = f_k(\Delta_k) - 1 \approx \left(1 - \varepsilon_k^{-1}\right) \Delta_k$. Using $\left(1 - \varepsilon_k^{-1}\right) HHI_k = f_k(\Delta_k) - 1$ or $HHI_k = \Delta_k$ to calibrate the value of Δ_k does not significantly affects the results.

[27] See, for example, Foster et al. (2008) or Castro et al. (2015).

Table 2: Baseline Calibration

Parameters	Value	Description	Target/Source
ε_k	5	substitution across firms	Monop Markup 1.25
N	389	# of sectors	BEA
N_k	578	median # firms in a sector	Census data
$\sum_k N_k$	5 576 852	total # of firms	Census data
Δ_k	0.037	median Pdty Herfindahl	sales HHI of the Census
β_k	0.027	median HH consumption share (%)	BEA
γ_k	55.90	median labor share (%)	BEA
Ω	2.19	I-O network density (when links> 1%, %)	BEA
a_k, c_k	0.34,0.30	median Firm-level pdty process	$\sigma_k = 0.1$ and Δ_k

NOTE: The first column gives the notation of the parameter in the model. The third column gives the description of the value in the second column. The fourth column is the data source or the associated calibration target.

The median number of firms in each sector is 578, and the total number of firms is equal to almost 5.6 millions. The median value of Δ_k across sectors is 0.037, which implies a value of the HHI of 0.063. Note that merger law starts to apply in the United States for a value of the HHI over 0.18. The value of the median markup is about 1.27. Under monopolistic competition, this markup would be 1.25 for a value of $\varepsilon_k = 5$. In this calibration, the median sector is relatively close to a sector under monopolistic competition, and it reflects the conservatism of the baseline calibration. Finally, the I-O network has a density of 2.19% and is very sparse; that is, 2.19% of all the possible $N^2 = 151321$ links have a value higher than 1%.

6.2 Aggregate and Sector-Level Volatility

For each of the 5.6 million firms, I simulate a path of productivity of $4,000$ periods.[28] To do so, I use Proposition 6 and simulate the law of motion of the productivity distribution for each sector. As in Carvalho and Grassi (2017), I follow the number of firms in each productivity bin rather than following the path of each firm. Doing so considerably reduces the computation cost of simulations. Note that even if Assumption 2 is key for this calibration strategy, by allowing the mapping between the HHI and the productivity concentration measure Δ_k, this assumption is not necessary to solve for the equilibrium allocation given the firm-productivity distribution in each period. Therefore, for each period t, I solve for the full problem at the firm level from which I recover sector-level markups and productivities that I aggregate in Y_t using the Proposition 8.[29] For this $4,000$-period time series, I compute aggregate volatility measured by the standard deviation of the percentage deviation of aggregate output Y_t. In Table 3, I report the standard deviation of the labor income and of the labor share, that is, the "downstream" and "upstream" parts of aggregate output:

[28] I simulate $5,000$ periods and drop the first $1,000$ periods.

[29] Rather than solving the firm-level problem for each firm, I solve this problem for each productivity bin because the firms in a bins are perfectly homogeneous. More details can be found in the numerical Appendix C.

Table 3: Aggregate Volatility

X_t	Total	Downstream	Upstream
$\sigma_X = sd\left[\log X_t\right]$	0.62	0.58	0.07
$\sigma_X / sd\left[\log Y_t\right]$	100	93.55	11.29

NOTE: The first row is the standard deviation of the percentage deviation of aggregate output Y_t, of the "downstream" and the "upstream" parts of aggregate output as defined in Section 4. The second row is the relative standard deviation of the "downstream" and the "upstream" parts of aggregate output with respect to aggregate volatility. Numbers are reported in percentage points. These statistics come from a $4,000$-periods simulation.

$$\log Y_t = \underbrace{\log w_t L}_{\text{labor income, "downstream"}} - \underbrace{\log \frac{w_t L}{P^C Y_t}}_{\text{labor share, "upstream"}}, \tag{10}$$

The first result is that the standard deviation of aggregate output Y_t is 0.62%. The same number in the Fernald (2014) data is 1.83%.[30] So the aggregate volatility in this model is $0.62/1.83 = 33.88\%$ of the aggregate volatility observed in the data. Note that in this model, this aggregate volatility arises purely from 5.6 million independent firm-level shocks. The reason this number is quantitatively non-negligible is that the central limit theorem and the "diversification argument" introduced by Lucas (1977) fail to apply. The first reason the central limit argument fails to apply is that the diversification across firms within a sector is weak. Indeed, within a sector, large firms represent a disproportionate market share as observed in the US Census Bureau concentration data. The "granular hypothesis" introduced by Gabaix (2011) is at play: shocks to these large firms do not average out. Following a shock to one of these large firms, another shock of the opposite sign is unlikely to hit another large firm and mitigate the first one. The second reason the central limit argument is not applying is that the diversification across the 389 sectors is governed by the "small-world" I-O network where a handful of highly connected hub-like sectors exist. As Acemoglu et al. (2012) and Carvalho (2010) show, diversification across sectors is weaker than without such an I-O network and translates into aggregate volatility.

The second result is that the volatility of the labor income, namely, the "downstream" part of aggregate output, is 93.55% of the aggregate volatility, whereas the volatility of the labor share, namely, the "upstream" part of aggregate output, is 11.29% of the aggregate volatility. In an economy without oligopolistic competition, all the aggregate volatility would be due to the downstream part of aggregate output, because the labor share, namely, the "upstream" part would be constant. This table illustrates the importance of the role of the propagation of changes in profit share and competition intensity following firm-level shocks. Consistent with Section 4's results of Propositions 9 and 10, the fact that the contribution of the "downstream" part of aggregate volatility is reduced compared to the monopolistic-competition case indicates aggregate fluctuations are led by shocks to large firms in their sectors. Following a shock to one of these large firms, the reduction in sector-level price that

[30] This number is the standard deviation of the percentage deviation of aggregate output from an HP trend as it is computed in Carvalho and Grassi (2017).

Table 4: Relative Contribution to Aggregate Volatility

X_t	"downstream"		"upstream"
	wage monop.	com. intensity	
σ_X/σ_Y	93.55		11.29
σ_X/σ_Y	101.48	15.67	11.29
$\sigma_X/\sigma_Y(Monop)$	100	0	0

NOTE: The first row is the ratio of the standard deviation of the "downstream" and "upstream" parts of aggregate output with the aggregate volatility. The second row shows the relative standard deviations of the (log) wage under monopolistic competition, of the difference of the (log) wage under oligopolistic and monopolistic competition, and of the "upstream" part of aggregate output. The third row is the same as the second row but under monopolistic competition. Numbers are reported in percentage points. These statistics comes from a $4,000$-period simulation.

propagates to downstream sectors is smaller because some of the gain in productivity is captured by an increase in markup. If, instead, the downstream propagation were higher than under monopolistic competition, the implication would be that fluctuations were led by shocks to medium-size firms. Indeed, a positive shock to one of these firms translates into a larger drop in the sector's price because some of this productivity gain also reduces the sector's markup and thus strengthens the downstream propagation.

Let us further decompose the labor income, that is, the "downstream" part of aggregate output, into the contribution of the wage under monopolistic competition and the contribution of the competition intensity:

$$\log Y_t^d = \log w_t L = \underbrace{\log w_t^{monop}}_{\text{wage under monopolistic}} + \underbrace{\log w_t - \log w_t^{monop}}_{\text{competition intensity}}. \tag{11}$$

Under Assumption 2, Equation 5 and Proposition 5 show that under monopolistic competition, the wage and the labor income, w_t^{monop}, at time t are entirely determined by $\overline{Z_{t,k}}$, whereas the term $\log w_t - \log w_t^{monop} = -\beta'(I-\Omega)^{-1}\{\log f_k(\Delta_{t,k})^{\frac{1}{\varepsilon_k - 1}}\}_k$ is entirely determined by the competition intensity measured by the statistic $\Delta_{t,k}$. From the expression of the elasticity of the "downstream" part of aggregate output in Equation 8, the interpretation of this decomposition is even clearer: the first term is as if the sector-level markup were assumed to be fixed, whereas the second term is the change in the sector-level markup. Table 4 shows the decomposition of the "downstream" part of aggregate output.

The first row of Table 4 reproduces the relative standard deviation of the "downstream" and "upstream" of Table 3. In the second row, I report the relative standard deviation of each terms of the righ-hand side of Equation 11, and in the last row, I report these latter numbers under monopolistic competition. Under monopolistic competition, all the volatility of aggregate output is due to change in the sum of productivity, whereas change in productivity concentration would have no effect. The volatility of aggregate output under monopolistic competition is 0.63 or 34.43% of the observed aggregate volatility in the data. From an aggregate perspective, the aggregate volatility under monopolistic and oligopolistic competition looks similar. However, as Table 4 shows, the propaga-

Table 5: Sector-level Volatility

$X_{k,t}$		$\log\left(\frac{w}{P_{k,t}}\right)^{(Monop)}$	$\log\frac{w}{P_{k,t}} - \log\left(\frac{w}{P_{k,t}}\right)^{(Monop)}$	$\log\frac{P_kY_k}{P^CC}$	$-\log\frac{wL}{P^CC}$
$\frac{\sigma_{X_k}}{\sigma_{Y_k}}$	$p10$	100.30	3.54	0.09	1.11
	$p50$	105.75	8.90	1.92	1.87
	$p90$	117.26	23.74	8.71	8.00
$\frac{\sigma_{X_k}}{\sigma_{Y_k}}$ (Monop)	$p10$	100	0	0	0
	$p50$	100	0	0	0
	$p90$	100	0	0	0

NOTE: The first three rows are the 10^{th}, 50^{th}, and 90^{th} across sectors of the percentile of the ratio of the standard deviation of each terms in the righ-hand side of Equation 12 and of the standard deviation of the (log) sector-level output $Y_{k,t}$. The last three rows report the same numbers under monopolistic competition. Numbers are reported in percentage points. These statistics comes from a $4,000$-period simulation.

tion patterns are entirely different. Taking oligopolistic competition into account, the propagation of changes in productivity of one firm to the downstream sectors is dampened by the response of the competition intensity. Furthermore, the latter also propagates to upstream sectors. The difference in propagation patterns between oligopolistic and monopolistic competition can be seen at the sector level. Let us decompose the sector-level output as:

$$\log Y_{k,t} = \log\left(\frac{w_t}{P_{k,t}}\right)^{Monop} + \left(\log\frac{w_t}{P_{k,t}} - \log\left(\frac{w_t}{P_{k,t}}\right)^{Monop}\right) + \log\frac{P_{k,t}Y_{k,t}}{P_t^CY_t} - \log\frac{w_tL}{P_t^CY_t}. \tag{12}$$

Under Assumption 2, the second term of the right-hand side of Equation 12, $\frac{w_t}{P_{k,t}} - \log\left(\frac{w_t}{P_{k,t}}\right)^{Monop}$, is only a function of the concentration measure $\Delta_{l,t}$.[31] Therefore, this term measures the role of oligopolistic competition. Furthermore, the last two terms in the right-hand side of Equation 12 are entirely determined by parameters and the sector-level markups (see Equations 3 and 6). Table 5 shows percentiles across sectors of the ratio of the standard deviation of each term in the righ-hand side of Equation 12 and of the standard deviation of the (log) sector-level output $Y_{k,t}$.

The first result is that the median relative standard deviation of the first and second terms in Equation 12 are, respectively, 105.75% and 8.90%. According to Proposition 11, these terms are due to the downstream propagation of changes in sector-level prices. The median relative standard deviation of the (log) sales share, $\log\frac{P_kY_k}{P^CC}$, is 1.92%. As Proposition 12 show, these fluctuations in the sales share are entirely due to the upstream propagation of change in sector-level profit share. The second result, is that these relative volatilities are heterogeneous across sectors. In 10% of the sectors, the volatility due to the propagation upstream of change in profit share represents more than 8.71% of the sector-level output volatility, whereas in 10% of the sectors, this number is almost negligible. The last result is that the propagation patterns across sectors under monopolistic competition are entirely different from the pattern under monopolistic competition, as the Table 5 shows.

[31] Combining Equation 2, the results in Proposition 12, and Equation 5 shows this results.

7 Conclusion

In this paper, I characterize the structural importance of a firm by its size, the role its sector plays in the I-O network, and by its sector's market structure. I highlight the role played by the interaction between the I-O linkages and the industrial organization both theoretically and numerically. The propagation of changes in profit share turns out to be important.

This paper also relates to the important literature on the micro-origin of aggregate fluctuations by addressing its internal inconsistency and providing a new quantification. Indeed, in this paper, large firms take into account the effect of their decisions on their sector price and output. Previous papers maintain the atomistic behavior assumption while studying the role played by a finite number of production units. Furthermore, the quantification provided here combined the "granular" and the "network" origin of aggregate fluctuations while allowing for a flexible market structure. Oligopolistic competition, compared to the monopolistic-competition benchmark, has little effect on the aggregate volatility arising from purely idiosyncratic shocks, but the propagation patterns are entirely different. A new propagation channel of productivity shocks arises through the endogenous response of markups and profit shares. The downstream propagation of firm-level productivity shocks is dampened while also propagating to upstream sectors. The interaction of oligopolistic competition and the I-O network is key for the latter.

This paper is also a starting point for studying the aggregate consequence of an increase in concentration. The framework presented here allows for the aggregation of the change in concentration and traces it back to a change in the concentration of firm-level productivity. Furthermore, if the concentration of firm-level productivity could be affected by policy, such as merger law, the model would be helpful in understanding the impact of such policy on the whole economy by taking into account the I-O network. I leave these subjects for future research.

References

Acemoglu, Daron, Ufuk Akcigit, and William Kerr, "Networks and the Macroeconomy: An Empirical Exploration," in "NBER Macroeconomics Annual 2015, Volume 30," University of Chicago Press, June 2015, pp. 276–335.

_ , Vasco M. Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, "The Network Origins of Aggregate Fluctuations," *Econometrica*, 09 2012, *80* (5), 1977–2016.

Aghion, Philippe, Ufuk Akcigit, and Peter Howitt, "What Do We Learn From Schumpeterian Growth Theory?," in "Handbook of Economic Growth," Vol. 2 of *Handbook of Economic Growth*, Elsevier, 2014, chapter 0, pp. 515–563.

Atkeson, Andrew and Ariel Burstein, "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, December 2008, *98* (5), 1998–2031.

Autor, David, David Dorn, Lawrence F. Katz, Christina Patterson, and John Van Reenen, "The Fall of the Labor Share and the Rise of Superstar Firms," Working Paper 23396, National Bureau of Economic Research May 2017.

Bak, Per, Kan Chen, Jose Scheinkman, and Michael Woodford, "Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics," *Ricerche Economiche*, March 1993, *47* (1), 3–30.

Baqaee, David Rezza, "Cascading Failures in Production Networks," 2016.

— and Emmanuel Farhi, "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," Working Paper 23145, National Bureau of Economic Research February 2017.

— and — , "Productivity and Misallocation in General Equilibrium.," Working Paper 24007, National Bureau of Economic Research November 2017.

Barkai, Simcha, "Declining labor and capital shares," Technical Report, London Business School 2017.

Barrot, Jean-Noel and Julien Sauvagnat, "Input Specificity and the Propagation of Idiosyncratic Shocks in Production Networks," *The Quarterly Journal of Economics*, 2016, *131* (3), 1543–1592.

Basu, Susanto, "Intermediate Goods and Business Cycles: Implications for Productivity and Welfare," *The American Economic Review*, 1995, *85* (3), 512–531.

— and John G. Fernald, "Aggregate productivity and aggregate technology," *European Economic Review*, 2002, *46* (6), 963 – 991. ISOM.

Bigio, Saki and Jennifer La'O, "Financial Frictions in Production Networks," Working Paper 22212, National Bureau of Economic Research April 2016.

Bilbiie, Florin O., Fabio Ghironi, and Marc J. Melitz, "Endogenous Entry, Product Variety, and Business Cycles," *Journal of Political Economy*, 2012, *120* (2), 304 – 345.

Boehm, Christoph E., Aaron Flaaen, and Nitya Pandalai-Nayar, "Input Linkages and the Transmission of Shocks: Firm-Level Evidence from the 2011 Tohoku Earthquake," 2016.

Carvalho, Vasco M, "Aggregate Fluctuations and the Network Structure of Intersectoral Trade," mimeo 2010.

Carvalho, Vasco M., "From Micro to Macro via Production Networks," *Journal of Economic Perspectives*, Fall 2014, *28* (4), 23–48.

— and Basile Grassi, "Large Firm Dynamics and the Business Cycle," May 2017.

— , Makoto Nirei, Yukiko Saito, and Alireza Tahbaz-Salehi, "Supply Chain Disruptions: Evidence from the Great East Japan Earthquake," 2016.

Castro, Rui, Gian Luca Clementi, and Yoonsoo Lee, "Cross Sectoral Variation in the Volatility of Plant Level Idiosyncratic Shocks," *The Journal of Industrial Economics*, 2015, *63* (1), 1–29.

Comin, Diego A. and Thomas Philippon, "The Rise in Firm-Level Volatility: Causes and Consequences," in "NBER Macroeconomics Annual 2005, Volume 20" NBER Chapters, National Bureau of Economic Research, Inc, October 2006, pp. 167–228.

Comin, Diego and Sunil Mulani, "Diverging Trends in Aggregate and Firm Volatility," *The Review of Economics and Statistics*, May 2006, *88* (2), 374–383.

Córdoba, Juan Carlos, "A Generalized Gibrat's Law," *International Economic Review*, November 2008, *49* (4), 1463–1468.

di Giovanni, Julian, Andrei A. Levchenko, and Isabelle Mejean, "Firms, Destinations, and Aggregate Fluctuations," *Econometrica*, 07 2014, *82* (4), 1303–1340.

Dixit, Avinash K and Joseph E Stiglitz, "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, June 1977, *67* (3), 297–308.

Durlauf, Steven N., "Nonergodic Economic Growth," *The Review of Economic Studies*, 1993, *60* (2), 349–366.

Fernald, John, "A quarterly, utilization-adjusted series on total factor productivity," Working Paper Series 2012-19, Federal Reserve Bank of San Francisco 2014.

Fort, Teresa C, John Haltiwanger, Ron S Jarmin, and Javier Miranda, "How Firms Respond to Business Cycles: The Role of Firm Age and Firm Size," *IMF Economic Review*, August 2013, *61* (3), 520–559.

Foster, Lucia, John Haltiwanger, and Chad Syverson, "Reallocation, Firm Turnover, and Efficiency: Selection on Productivity or Profitability?," *American Economic Review*, March 2008, *98* (1), 394–425.

Gabaix, Xavier, "The Granular Origins of Aggregate Fluctuations," *Econometrica*, 05 2011, *79* (3), 733–772.

Gibrat, Robert, *Les Inégalités économiques*, Paris, France: Librairie du Recueil Sirey, 1931.

Hannah, L. and J.A. Kay, *Concentration in modern industry: theory, measurement, and the U.K. experience*, Macmillan, 1977.

Hopenhayn, Hugo A, "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, September 1992, *60* (5), 1127–50.

Hopenhayn, Hugo A. and Edward C. Prescott, "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," *Econometrica*, 1992, *60* (6), 1387–1406.

Hulten, Charles R., "Growth accounting with intermediate inputs," *The Review of Economic Studies*, 1978, pp. 511–518.

Imbs, Jean and Isabelle Mejean, "Elasticity Optimism," *American Economic Journal: Macroeconomics*, July 2015, *7* (3), 43–83.

Jones, Charles I., "Intermediate Goods and Weak Links in the Theory of Economic Development," *American Economic Journal: Macroeconomics*, April 2011, *3* (2), 1–28.

— , "Misallocation, Economic Growth, and Input–Output Economics," in Daron Acemoglu, Manuel Arellano, and EddieEditors Dekel, eds., *Advances in Economics and Econometrics: Tenth World Congress*, Vol. 2 of *Econometric Society Monographs*, Cambridge University Press, 2013, p. 419–456.

Jovanovic, Boyan, "Micro Shocks and Aggregate Risk," *The Quarterly Journal of Economics*, May 1987, *102* (2), 395–409.

Kehrig, Matthias and Nicolas Vincent, "Growing Productivity Without Growing Wages: The Micro-Level Anatomy of the Aggregate Labor Share Decline," Technical Report Working Paper No. 244, Economic Research Initiatives at Duke (ERID) April 2017.

Krugman, Paul R., "Increasing returns, monopolistic competition, and international trade," *Journal of International Economics*, November 1979, *9* (4), 469–479.

Loecker, Jan De and Jan Eeckhout, "The Rise of Market Power and the Macroeconomic Implications," Working Paper 23687, National Bureau of Economic Research August 2017.

Long, John B. Jr and Charles I Plosser, "Real Business Cycles," *Journal of Political Economy*, February 1983, *91* (1), 39–69.

Lucas, Robert E, "Understanding business cycles," *Carnegie-Rochester Conference Series on Public Policy*, 1977, *5* (Supplement C), 7 – 29.

Magerman, Glenn, Karolien De Bruyne, Emmanuel Dhyne, and Jan Van Hove, "Heterogeneous Firms and the Micro Origins of Aggregate Fluctuations," 2016.

Melitz, Marc J. and Gianmarco I. P. Ottaviano, "Market Size, Trade, and Productivity," *Review of Economic Studies*, 2008, *75* (1), 295–316.

Ottaviano, Gianmarco, Takatoshi Tabuchi, and Jacques-François Thisse, "Agglomeration and Trade Revisited," *International Economic Review*, May 2002, *43* (2), 409–436.

Severini, Thomas A., *Elements of Distribution Theory*, Cambridge University Press, 2005.

Sutton, John, "Gibrat's Legacy," *Journal of Economic Literature*, 1997, *35* (1), 40–59.

Yeh, Chen, "Are firm-level idiosyncratic shocks important for U.S. aggregate volatility?," 2017.

Zhelobodko, Evgeny, Sergey Kokovin, Mathieu Parenti, and Jacques-François Thisse, "Monopolistic Competition: Beyond the Constant Elasticity of Substitution," *Econometrica*, November 2012, *80* (6), 2765–2784.

Appendix to "IO in I-O: Size, Industrial Organization and the Input-Output Network Make a Firm Structurally Important"

Basile Grassi

A Proof Appendix

A.1 Proof of Proposition 2 (Firm's Approximation)

The first step is to rewrite the system of equations of Proposition 1 in terms of the following *perceived* elasticity of demand: $\varepsilon(k,i) = \frac{\mu(k,i)}{\mu(k,i)-1}$. Then let us define the following system of equations for a given parameter χ:

$$P(k,i) = \frac{\varepsilon(k,i)}{\varepsilon(k,i)-1}\lambda(k,i)$$
$$s(k,i) = \frac{P(k,i)y(k,i)}{P_k Y_k} = \left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k}$$
$$\varepsilon(k,i) = \begin{cases} \varepsilon_k & \text{Under Monopolistic Competition} \\ \varepsilon_k - \chi(\varepsilon_k - 1)s(k,i) & \text{Under Bertrand Competition} \\ \left(\frac{1}{\varepsilon_k} + \chi(1-\frac{1}{\varepsilon_k})s(k,i)\right)^{-1} & \text{Under Cournot Competition} \end{cases}$$

When $\chi = 1$, the above system is exactly the one described in Proposition 1 in terms of $\varepsilon(k,i)$. When $\chi = 0$, both the Bertrand and Cournot cases reduce to the monopolistic case. I now focuse on these two cases.

Let us reduce the above system of equations to one equation determining the sales share $s(k,i)$ of the firm i in sector k by subsituting the expression of $\varepsilon(i,k)$ and $P(k,i)$:

$$s(k,i) = \begin{cases} \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k-1)s(k,i)}\right)^{\varepsilon_k-1}\left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k} & \text{Under Bertrand} \\ \left(1-\frac{1}{\varepsilon_k} - \chi(1-\frac{1}{\varepsilon_k})s(k,i)\right)^{\varepsilon_k-1}\left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k} & \text{Under Cournot} \end{cases}$$

Let us rewrite the above equation using the unknown $X(\omega,\chi) = s(k,i)$ with $\omega = \left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$ and by the function $\mathcal{H}(X,\omega,\chi)$ such that:

$$\mathcal{F}(\omega,\chi) = \mathcal{H}(X(\omega,\chi),\omega,\chi) = 0 \tag{13}$$

with

$$\mathcal{H}(X,\omega,\chi) = \begin{cases} X - \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k-1)X}\right)^{(\varepsilon_k-1)}\omega & \text{Under Bertrand} \\ X - \left(1-\frac{1}{\varepsilon_k} - \chi(1-\frac{1}{\varepsilon_k})X\right)^{(\varepsilon_k-1)}\omega & \text{Under Cournot.} \end{cases}$$

As explained earlier, $X(\omega,0) = \widehat{s}(k,i)$ is the solution under monopolistic competition. The solution of this system $X(\omega,\chi)$ satisfies at the second order:

$$X(\omega,\chi) = X(\omega,0) + \chi X'(\omega,0) + \chi^2 X''(\omega,0) + o(\chi^2),$$

where $X'(\omega,\chi) := \frac{\partial X}{\partial\chi}(\omega,\chi)$ and $X''(\omega,\chi) := \frac{\partial X'}{\partial\chi}(\omega,\chi)$.

For $\chi = 1$, it yields an approximation of the solution for the Oligopolistic case:

$$X(\omega,1) \approx X(\omega,0) + X'(\omega,0) + X''(\omega,0).$$

Let us compute these derivatives by differentiating Equation 13:

$$\mathcal{F}'_\chi(\omega,\chi) = 0 = X'(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + \mathcal{H}'_\chi(X(\omega,\chi),\omega,\chi)$$
$$\mathcal{F}''_\chi(\omega,\chi) = 0 = X''(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + (X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi),$$

from which it follows

$$X'(\omega,\chi) = -\frac{\mathcal{H}'_\chi(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_X(X(\omega,\chi),\omega,\chi)}$$

$$X''(\omega,\chi) = -\frac{(X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_X(X(\omega,\chi),\omega,\chi)},$$

and evaluating this at $(\omega, 0)$:

$$X'(\omega,0) = -\frac{\mathcal{H}'_\chi(X(\omega,0),\omega,0)}{\mathcal{H}'_X(X(\omega,0),\omega,0)}$$

$$X''(\omega,0) = -\frac{(X'(\omega,0))^2\mathcal{H}''_{XX}(X(\omega,0),\omega,0) + 2X'(\omega,0)\mathcal{H}''_{\chi X}(X(\omega,0),\omega,0)}{\mathcal{H}'_X(X(\omega,0),\omega,0)},$$

We are left to compute the derivative of $\mathcal{H}(X,\omega,\chi)$ and substitute, which yields

$$X'(\omega,0) = \begin{cases} -(1-\frac{1}{\varepsilon_k})X(\omega,0)^2 & \text{Under Bertrand} \\ -(\varepsilon_k - 1)X(\omega,0)^2 & \text{Under Cournot} \end{cases}$$

$$X''(\omega,0) = \begin{cases} (1-\frac{1}{\varepsilon_k})^2(1-\frac{1}{\varepsilon_k-1})X(\omega,0)^3 & \text{Under Bertrand} \\ (\varepsilon_k - 1)^2(3-\frac{1}{\varepsilon_k-1})X(\omega,0)^3 & \text{Under Cournot.} \end{cases}$$

which yields

$$X(\omega,1) \approx \begin{cases} X(\omega,0)\left(1-(1-\frac{1}{\varepsilon_k})X(\omega,0) + (1-\frac{1}{\varepsilon_k})^2(1-\frac{1}{\varepsilon_k-1})X(\omega,0)^2\right) & \text{Under Bertrand} \\ X(\omega,0)\left(1-(\varepsilon_k-1)X(\omega,0) + (\varepsilon_k-1)^2(3-\frac{1}{\varepsilon_k-1})X(\omega,0)^2\right) & \text{Under Cournot} \end{cases}$$

By substituting $X(\omega,1) = s(k,i)$ and $X(\omega,0) = \widehat{s}(k,i)$, we get the result. □

A.2 Proof of Proposition 4 (Sector-Level Markup)

To prove this proposition, I substitute the result of Proposition 1 into Equation 1, reproduced here for convenience:

$$\mu_k = \left(\sum_{i=1}^{N_k}\mu(k,i)^{-1}s(k,i)\right)^{-1}. \tag{1}$$

Let us first focus on the monopolistic competition case, and then turn to the Cournot and Bertrand cases.

Monopolistic case: Let us first look at the monopolistic competition case in which markups charged by firms in sector k are identical and equal to $\frac{\varepsilon_k}{\varepsilon_k-1}$. Substituting $\mu(k,i)$ into Equation 1 leads to $\mu_k = \frac{\varepsilon_k}{\varepsilon_k-1}\left(\sum_{i=1}^{N_k}s(k,i)\right)^{-1} = \frac{\varepsilon_k}{\varepsilon_k-1}$ because the sum of the sales share of firms in sector k is equal to 1.

Cournot case: In this case, the markup charged by firm i in sector k is equal to $\mu(k,i) = \frac{\varepsilon_k}{\varepsilon_k-1-(\varepsilon_k-1)s(k,i)}$. Let us substitute it into Equation 1. After some simplification, we have:

$$\mu_k^{-1} = \frac{\varepsilon_k-1}{\varepsilon_k}\sum_{i=1}^{N_k}s(k,i) - \frac{\varepsilon_k-1}{\varepsilon_k}\sum_{i=1}^{N_k}s(k,i)^2 = \frac{\varepsilon_k-1}{\varepsilon_k}\left(1-\sum_{i=1}^{N_k}s(k,i)^2\right),$$

where the last equality comes from the fact that $\sum_{i=1}^{N_k}s(k,i) = 1$.

Bertrand case: In this case, the markup charged by firm i in sector k is equal to $\mu(k,i) = \frac{\varepsilon_k-(\varepsilon_k-1)s(k,i)}{\varepsilon_k-1-(\varepsilon_k-1)s(k,i)}$. Let us substitute it in Equation 1. After some simplification we have:

$$\mu_k^{-1} = 1 - \frac{1}{\varepsilon_k}\sum_{i=1}^{N_k}s(k,i)\frac{1}{1-\frac{\varepsilon_k-1}{\varepsilon_k}s(k,i)}$$

Note that because $\varepsilon_k > 1$ and $s(k,i) < 1$, we have $0 < \frac{\varepsilon_k-1}{\varepsilon_k}s(k,i) < 1$. We can expand the series and therefore

$\frac{1}{1-\frac{\varepsilon_k-1}{\varepsilon_k}s(k,i)} = \sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m s(k,i)^m$. After substituting in the previous equation, we get

$$\begin{aligned}
\mu_k^{-1} =& 1 - \frac{1}{\varepsilon_k}\sum_{i=1}^{N_k}\sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m s(k,i)^{m+1} \\
=& 1 - \frac{1}{\varepsilon_k}\sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m \sum_{i=1}^{N_k} s(k,i)^{m+1} \\
=& 1 - \frac{1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\sum_{m=1}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m \sum_{i=1}^{N_k} s(k,i)^{m+1} \\
=& \frac{\varepsilon_k-1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k-1}\sum_{m=2}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^{m-1} \sum_{i=1}^{N_k} s(k,i)^{m}\right),
\end{aligned}$$

where for the first to the second line, I use the fact that the sum over the firms in sector k index by i is finite, where for the second to the third line, I take out the first term of the sum over the index m, and where the last line comes from rearranging terms and reindexing of the sum over m. □

A.3 Proof of Proposition 5 (Sector Allocation)

The structure of the proof is as follows. First, I find the relationship (Equation 2) between sector prices and sector level productivity and markup. Second, I show Lemmas 1 and 2, which relate the sector-level productivity and markup to other sector prices and the wage. Finally, I combine these results to solve for the sector allocation.

A.3.1 Proof of Equation 2:

As show earlier in Section 3.2, the sector-level marginal cost and markup are such that $\lambda_k = \mu_k^{-1}P_k$. Using the fact that $\lambda_k = Z_k^{-\gamma_k}w^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}}$, we have

$$P_k = \mu_k\left(\frac{w}{Z_k}\right)^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}} \quad \text{and} \quad \log P_k = \log\mu_k\left(\frac{w}{Z_k}\right)^{\gamma_k} + \sum_{l=1}^{N}\omega_{k,l}\log P_l \tag{14}$$

Rewriting the last equation in matrix form yields

$$\{\log P_k\}_k = \left\{\log\mu_k\left(\frac{w}{Z_k}\right)^{\gamma_k}\right\}_k + \Omega\{\log P_k\}_k$$

where the $N\times N$ matrix Ω is such that $\Omega = \{\omega_{k,l}\}_{1\leq k,l\leq N}$. Finally, Equation 2 comes from the pre-multiplication of the following expression by the matrix $(I-\Omega)^{-1}$:

$$(I-\Omega)\{\log P_k\}_k = \left\{\log\mu_k\left(\frac{w}{Z_k}\right)^{\gamma_k}\right\}_k$$

□

A.3.2 Proof of Equation 3:

The market-clearing condition for the variety i of sector k's good is such that the supply is equal to the demand from the household and from other firms in the economy:

$$P(k,i)y(k,i) = P(k,i)c(k,i) + \sum_{l=1}^{N}\sum_{j=1}^{N_l}P(k,i)x(l,j,k,i),$$

where $c(k,i)$ is the demand of variety i of sector k's good by the household and $x(l,j,k,i)$ is the demand of variety i of sector k's good from firm j in sector l. The household's problem gives $P(k,i)c(k,i) =$

$\beta_k\left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k}P^CC$, whereas the cost-minimization problem of firm j in sector l gives $P(k,i)x(l,j,k,i) = \omega_{l,k}\left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k}\lambda(l,j)y(l,j)$. Summing over the firms in sector k and using the fact that $P_k=\sum_{i=1}^{N_k}P(k,i)^{1-\varepsilon_k}$, we have

$$P_kY_k=\sum_{i=1}^{N_k}p(k,i)y(k,i)=\beta_kP^CC+\sum_{l=1}^{N}\omega_{l,k}\sum_{j=1}^{N_l}\lambda(l,j)y(l,j)=\beta_kP^CC+\sum_{l=1}^{N}\omega_{l,k}\mu_l^{-1}P_lY_l$$

where in the last equality, I use the definition of the sector marginal cost λ_l and the fact that $\lambda_l=\mu_l^{-1}P_l$. Let us define the $N\times N$ matrix $\widetilde{\Omega}=\{\mu_k^{-1}\omega_{k,l}\}_{k,l}$. The above equation in vector form yields

$$\left\{\frac{P_kY_k}{P^CC}\right\}_k'=\beta'+\left\{\frac{P_lY_l}{P^CC}\right\}_l'\widetilde{\Omega}\quad\Rightarrow\quad\left\{\frac{P_kY_k}{P^CC}\right\}_k'=\beta'(I-\widetilde{\Omega})^{-1}.$$

□

A.3.3 Two Lemmas:

Let us first prove two lemmas that simplifythe expression of sectors' productivity and markup.

Lemma 1 ***(Productivity)*** *Under Assumption 2, the sector-level productivity Z_k satisfies*

$$Z_k^{-\gamma_k}=\begin{cases}X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}}\overline{Z_k} & \textit{Under Monopolistic Competition}\\ X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}}\left(\overline{Z_k}-X_k\overline{Z_k}^2\Delta_k\right) & \textit{Under Bertrand Competition}\\ X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}}\left(\overline{Z_k}-\varepsilon_kX_k\overline{Z_k}^2\Delta_k\right) & \textit{Under Cournot Competition}\end{cases}$$

where $X_k=\left(P_k^{-1}\frac{\varepsilon_k}{\varepsilon_k-1}w^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}}\right)^{(1-\varepsilon_k)}$ and where $\overline{Z_k}$ and Δ_k are defined in Section 3.1.

Proof of Lemma 1: Let us first look at the monopolistic case before turning to the Bertrand and Cournot cases.

Monopolistic case: Under monopolistic competition, firm i in sector k charges a (constant) markup $\frac{\varepsilon_k}{\varepsilon_k-1}$ over its marginal cost $\lambda(k,i)$. Note the firm-level marginal cost is equal to

$$\lambda(k,i)=Z(k,i)^{-\gamma_k}w^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}}$$

It follows that

$$\frac{y(k,i)}{Y_k}=\left(\frac{P(k,i)}{P_k}\right)^{-\varepsilon_k}=\left(\frac{\varepsilon_k}{\varepsilon_k-1}P_k^{-1}\lambda(k,i)\right)^{-\varepsilon_k}=Z(k,i)^{\gamma_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}P_k^{-1}w^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}}\right)^{-\varepsilon_k}.$$

Substituting the above expression into the expression of Z_k yields

$$Z_k^{-\gamma_k}=\sum_{i=1}^{N_k}Z(k,i)^{-\gamma_k}\frac{y(k,i)}{Y_k}=\left(\frac{\varepsilon_k}{\varepsilon_k-1}P_k^{-1}w^{\gamma_k}\prod_{l=1}^{N}P_l^{\omega_{k,l}}\right)^{-\varepsilon_k}\sum_{i=1}^{N_k}Z(k,i)^{\gamma_k(\varepsilon_k-1)}$$

which implies the result

$$Z_k^{-\gamma_k}=X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}}\sum_{i=1}^{N_k}Z(k,i)^{\gamma_k(\varepsilon_k-1)}=X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}}\overline{Z_k}$$

Cournot case: Let us first note that $\frac{y(k,i)}{Y_k}=P_kp(k,i)^{-1}s(k,i)=P_k\lambda(k,i)^{-1}\mu(k,i)^{-1}s(k,i)$. The sales share under monopolistic competition is $\hat{s}(k,i)=P_k^{\varepsilon_k-1}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k}\lambda(k,i)^{1-\varepsilon_k}$, whereas $\lambda(k,i)=$

$Z(k,i)^{-\gamma_k} X_k^{\frac{1}{1-\varepsilon_k}} P_k \frac{\varepsilon_k - 1}{\varepsilon_k}$. It follows that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$. Note also that $\lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}}$.

Under Cournot competition according to Proposition 1, we have

$$\frac{y(k,i)}{Y_k} = \lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} (s(k,i) - s(k,i)^2) = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - s(k,i)^2).$$

Under Assumption 2,the sales share of firm i in sector k satisfies $s(k,i) - s(k,i)^2 = \hat{s}(k,i) - \varepsilon_k \hat{s}(k,i)^2$. Equipped with all these expressions, let us look at

$$\begin{aligned}
Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k} &= \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - s(k,i)^2) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(\hat{s}(k,i) - \varepsilon_k \hat{s}(k,i)^2 \right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k - \varepsilon_k Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} X_k^2 \right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \left(X_k \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k(1-\varepsilon_k)} - \varepsilon_k X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} \right) \\
&= X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\overline{Z_k} - \varepsilon_k X_k \overline{Z_k}^2 \Delta_k \right).
\end{aligned}$$

Bertrand case:

Under Bertrand competition, according to Proposition 1, $\mu(k,i) = \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)}$, and therefore

$$\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} \frac{1 - s(k,i)}{1 - \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i)} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i)(1 - s(k,i)) \left(1 + \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i) + (\frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i))^2 \right),$$

where the last equality holds for a second-order approximation for $s(k,i) \longrightarrow 0$. At the second order, we thus have $\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2 \right)$. Using the fact that $\frac{y(k,i)}{Y_k} = P_k \lambda(k,i)^{-1} \mu(k,i)^{-1} s(k,i)$ and $\lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}}$, the output share of firm i in sector k is

$$\frac{y(k,i)}{Y_k} = \lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2 \right) = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2 \right).$$

Under Assumption 2,the sales share of firm i in sector k satisfies $s(k,i) - \varepsilon_k^{-1} s(k,i)^2 = \hat{s}(k,i) - \hat{s}(k,i)^2$. Equipped with all these expressions, let us look at

$$\begin{aligned}
Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k} &= \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - \varepsilon_k^{-1} s(k,i)^2) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(\hat{s}(k,i) - \hat{s}(k,i)^2 \right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k - Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} X_k^2 \right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \left(X_k \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k(1-\varepsilon_k)} - X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} \right) \\
&= X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\overline{Z_k} - X_k \overline{Z_k}^2 \Delta_k \right).
\end{aligned}$$

□

Lemma 2 ***(Markup)*** *Under Assumption 2, sector k's markup satisfies*

$$\mu_k^{-1} = \begin{cases} \frac{\varepsilon_k - 1}{\varepsilon_k} & \textit{Under Monopolistic Competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k} X_k^2 \overline{Z_k}^2 \Delta_k\right) & \textit{Under Bertrand Competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - X_k^2 \overline{Z_k}^2 \Delta_k\right) & \textit{Under Cournot Competition.} \end{cases}$$

where $X_k = \left(P_k^{-1} \frac{\varepsilon_k}{\varepsilon_k - 1} w^{\gamma_k} \prod_{l=1}^N P_l^{\omega_{k,l}}\right)^{(1-\varepsilon_k)}$ *and where* $\overline{Z_k}$ *and* Δ_k *are defined in Section 3.1.*

Proof of Lemma 2: Let us first look at the Cournot case before turning to the Bertrand case.

Cournot case: Proposition 4 shows that under Cournot competition, we have $\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \sum_{i=1}^{N_k} s(k,i)^2\right)$ whereas under Assumption 2, we have $s(k,i)^2 = \hat{s}(k,i)^2$. Using the fact that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$,

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{2\gamma_k(\varepsilon_k - 1)}\right).$$

Bertrand case: As shown in the proof of Lemma 1 under Bertrand competition, the markup and the sales share of firm i in sector k satisfy up to a second-order approximation:

$$\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right).$$

It follows that the sector-level markup,

$$\mu_k^{-1} = \sum_{i=1}^{N_k} \frac{\varepsilon_k - 1}{\varepsilon_k}\left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \varepsilon_k^{-1} \sum_{i=1}^{N_k} s(k,i)^2\right)$$

because $\sum_{i=1}^{N_k} s(k,i) = 1$. Under Assumption 2, we have that $s(k,i)^2 = \hat{s}(k,i)^2$. Using the fact that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$, the result follows. □

A.3.4 Proof of the Proposition 5:

In the last step of this proof, let us rewrite Equation 14 as

$$\mu_k Z_k^{-\gamma_k} = \frac{\varepsilon_k}{\varepsilon_k - 1}\left[\frac{\varepsilon_k - 1}{\varepsilon_k} P_k w^{-\gamma_k} \prod_{l=1}^N P_l^{-\omega_{k,l}}\right].$$

The term on the right-hand side in the brackets of the above equation is equal to $X_k^{\frac{1}{\varepsilon_k - 1}}$. We can then rewrite this equation as

$$Z_k^{-\gamma_k} = \mu_k^{-1} \frac{\varepsilon_k}{\varepsilon_k - 1} X_k^{\frac{1}{\varepsilon_k - 1}} \tag{15}$$

Bertrand case: Let us substitute in Equation 15, the expression of the productivity and the markup in Lemmas 1 and 2:

$$X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}}\left(\overline{Z_k} - X_k \overline{Z_k}^2 \Delta_k\right) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k} X_k^2 \overline{Z_k}^2 \Delta_k\right) \frac{\varepsilon_k}{\varepsilon_k - 1} X_k^{\frac{1}{\varepsilon_k - 1}}$$

Rearranging terms yields the following quadratic equation in unknown X_k:

$$(1 - \varepsilon_k^{-1}) \overline{Z_k}^2 \Delta_k X_k^2 - \overline{Z_k} X_k + 1 = 0. \tag{16}$$

First note that the monopolistic case is nested in the above equation (see Lemmas 1 and 2). Indeed, by taking $\Delta_k = 0$, we recover the solution of the above equation under monopolistic competition: $X_k = 1/\overline{Z_k}$. Second, Equation 16 admits a solution only if $\overline{Z_k}^2 - 4(1 - \varepsilon_k^{-1}) \overline{Z_k}^2 \Delta_k \geq 0$ or equivalently when $\Delta_k \leq \frac{1}{4(1 - \varepsilon_k^{-1})}$. In the case of strict inequality, this equation admits the following two solutions:

$$X_k^+ = \frac{1+\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}} \quad \text{and} \quad X_k^- = \frac{1-\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}}.$$

For $\Delta_k \longrightarrow 0$, we cano see $X_k^+ \longrightarrow \infty$ and $X_k^- \longrightarrow \frac{1}{\overline{Z_k}}$. To ensure continuity of the solutions with the monopolistic case, X_k^- is the only admissible solution, and therefore using the notation f_k of the Proposition 5,

$$X_k == \frac{1-\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}} = \frac{f_k(\Delta_k)}{\overline{Z_k}}.$$

Let us now solve for the productivity. Using Lemma 1, we have $X_k^{\frac{1}{\varepsilon_k-1}}\left(\overline{Z_k}X_k - X_k^2\overline{Z_k}^2\Delta_k\right)$. Using Equation 16, to see that $X_k^2\overline{Z_k}^2\Delta_k = \frac{\overline{Z_k}X_k-1}{1-\varepsilon_k^{-1}}$, the productivity in sector k satisfies

$$Z_k^{-\gamma_k} = X_k^{\frac{1}{\varepsilon_k-1}}\frac{\varepsilon_k-\overline{Z_k}X_k}{\varepsilon_k-1}.$$

The markup expression is also found using the same reasoning. Combining Lemma 2 and Equation 16 yields that $\mu_k^{-1} = \frac{\varepsilon_k-\overline{Z_k}X_k}{\varepsilon_k}$.

Cournot case: For the Cournot case, I follow the same logic. By combining Equation 15 and Lemmas 1 and 2, X_k is the solution of the following quadratic equation:

$$(\varepsilon_k-1)\overline{Z_k}^2\Delta_k X_k^2 - \overline{Z_k}X_k + 1 = 0. \tag{17}$$

This equation, for $\Delta_k < \frac{1}{4(\varepsilon_k-1)}$, has one admissible solution[32]:

$$X_k = \frac{1-\sqrt{1-4(\varepsilon_k-1)\Delta_k}}{2\Delta_k(\varepsilon_k-1)\overline{Z_k}} = \frac{f_k(\Delta_k)}{\overline{Z_k}}.$$

Using Equation 17 and Lemmas 1 and 2, it is easy to show that under Cournot competition

$$Z_k^{-\gamma_k} = X_k^{\frac{1}{\varepsilon_k-1}}\frac{\varepsilon_k-\overline{Z_k}X_k}{\varepsilon_k-1} \quad \text{and} \quad \mu_k^{-1} = \frac{\varepsilon_k-\overline{Z_k}X_k}{\varepsilon_k}.$$

□

A.4 Proof of Proposition 8 (Equilibrium Allocation)

Wage (Equation 5): Without loss of generality, let us normalize the composite consumption good to $P^C = 1$. It implies $0 = \log 1 = \log P^C = \sum_{k=1}^N \beta_k \log P_k = \beta' \{\log P_k\}_k$, where the last expression is an inner product of the two vectors β and $\{\log P_k\}_k$. In this last expression, let us subsitute the expression of sector-level price (Equation 2):

$$0 = \beta'(I-\Omega)^{-1}\left\{\log \mu_l \left(\frac{w}{Z_l}\right)^{\gamma_l}\right\}_l = \beta'(I-\Omega)^{-1}\{\gamma_l\}_l \log w + \beta'(I-\Omega)^{-1}\left\{\log \mu_l Z_l{}^{-\gamma_l}\right\}_l.$$

Note that $\Omega\mathbb{I} = \left\{\sum_{l=1}^N \omega_{k,l}\right\}_k = \{1-\gamma_k\}_k = \mathbb{I} - \{\gamma_k\}_k$, where $\mathbb{I} = \{1\}_k$ is the vector of ones. It implies $(I-\Omega)^{-1}\{\gamma_l\}_l = \mathbb{I}$. Furthermore, because $\sum_{k=1}^N \beta_k = 1$, it follows that $\beta'(I-\Omega)^{-1}\{\gamma_l\}_l = \beta'\mathbb{I} = \sum_{k=1}^N \beta_k = 1$. Using this last expression, we have the expression of the wage:

$$\log w = -\beta'(I-\Omega)^{-1}\left\{\log \mu_l Z_l{}^{-\gamma_l}\right\}_l.$$

□

[32]An admissible solution such that $X_k \underset{\Delta_k\to 0}{\longrightarrow} \frac{1}{\overline{Z_k}}$.

Aggregate profit share (Equation 6): From the firm's problem, it is clear that the profit $\pi(k,i)$ of firm i in sector k is such that $\pi(k,i) = P(k,i)y(k,i) - \lambda(k,i)y(k,i)$. Summing over the firms in sector k yields:

$$\pi_k = \sum_{i=1}^{N_k} P(k,i)y(k,i) - \sum_{i=1}^{N_k} \lambda(k,i)y(k,i) = P_k Y_k - \lambda_k Y_k = (1-\mu_k^{-1})P_k Y_k,$$

where I use the definition of the marginal cost in sector k and the fact that $\lambda_k = \mu_k^{-1} P_k$. Finally, aggregate profit is equal to the sum of the profit in each sector:

$$\frac{Pro}{P^C C} = \sum_{k=1}^{N} \pi_k \frac{\pi_k}{P^C C} = \sum_{k=1}^{N} (1-\mu_k^{-1}) \frac{P_k Y_k}{P^C C} = \left\{ \frac{P_k Y_k}{P^C C} \right\}_k' \{1-\mu_k^{-1}\}_k .$$

Substituting Equation 3 yields the result. □

Aggregate output (Equation 7): The household budget constraint is such that total expenditure is equal to the labor and profit income:

$$P^C C = wL + Pro \quad \Leftrightarrow \quad C = w + \frac{Pro}{P^C C} C,$$

where I use the normalization of the price $P^C = 1$ and of the labor $L = 1$. Note that in this framework, $Y = C$. Rearranging terms and taking logs gives the results. □

A.5 Proof of Propositions 9 and 10 (Elasticity of Aggregate Output)

Proposition 9: Combining the expression of the wage w in Proposition 8 and the expression of the sectoral markups and productivities under Assumption 2 given by Proposition 5, we have

$$\log w = -\overline{\beta}' \left\{ \log \frac{\varepsilon_k}{\varepsilon_k - 1} \left(\overline{Z_k}\right)^{\frac{-1}{\varepsilon_k - 1}} f_k(\Delta_k)^{\frac{1}{\varepsilon_k - 1}} \right\}_k = -\sum_{k=1}^{N} \overline{\beta}_k \left(\frac{-1}{\varepsilon_k - 1} \log \overline{Z_k} + \frac{1}{\varepsilon_k - 1} \log f_k(\Delta_k) + \log \frac{\varepsilon_k}{\varepsilon_k - 1} \right).$$

Taking derivative of the above expression with respect to $\log \overline{Z_k}$ and $\log \Delta_k$ yields

$$\frac{\partial \log w}{\partial \log \overline{Z_k}} = \frac{\overline{\beta}_k}{\varepsilon_k - 1} \quad \text{and} \quad \frac{\partial \log w}{\partial \log \Delta_k} = -\frac{\overline{\beta}_k}{\varepsilon_k - 1} e_k.$$

where e_k is the elasticity of f_k with respect to Δ_k: $e_k = \frac{d \log f_k(\Delta_k)}{d \log \Delta_k}$. Using the fact that $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{2}{\Delta_k} \left(\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} - \Delta_k \right) \frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$ and using the chain rule, we can compute the elasticity of the wage with respect to the productivity of firm i in sector k:

$$\begin{aligned}
\frac{\partial \log w}{\partial \log Z(k,i)} &= \frac{\partial \log w}{\partial \log \overline{Z_k}} \frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} + \frac{\partial \log w}{\partial \log \Delta_k} \frac{\partial \log \Delta_k}{\partial \log Z(k,i)} \\
&= \frac{\overline{\beta}_k}{\varepsilon_k - 1} \left(1 + \frac{2e_k}{\Delta_k} \left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} \right) \right) \frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}.
\end{aligned}$$

□

Proposition 10: Let us first prove a technical lemma that turns out to be useful, which compute the derivative of the sector-level sales share with respect to the inverse of the sector-level markups.

Lemma 3 ***(Sector-level sales-share derivative)*** *Under Assumption 2, the derivative of the vector of sector-level sales share $\widetilde{\beta} = \{\frac{P_k Y_k}{P^C C}\}_k$ with respect to μ_k^{-1} is*

$$\frac{\partial \widetilde{\beta}'}{\partial (\mu_k^{-1})} = \mu_k \widetilde{\beta}_k v_k' \left[(I - \widetilde{\Omega})^{-1} - I \right],$$

where v_k is the $(N \times 1)$ vector where all elements are zero except the k^{th}.

Proof of the lemma: Equation 3 in Proposition 5 tells us $\widetilde{\beta}' \equiv \{\frac{P_k Y_k}{P^C C}\}_k' = \beta'(I - \widetilde{\Omega})^{-1} = \beta'(I - S\Omega)^{-1}$, where

S is the diagonal matrix with the element of the vector $\{\mu_k^{-1}\}_k$ i.e $S = diag(\mu_k^{-1})$. Thanks to the fact that for a matrix A, the derivative of its inverse is $\frac{\partial A^{-1}}{\partial x} = -A^{-1}\frac{\partial A}{\partial x}A^{-1}$, we have

$$\frac{\partial \widetilde{\beta}'}{\partial \mu_k^{-1}} = -\beta'(I-\widetilde{\Omega})^{-1}\frac{\partial (I-S\Omega)}{\partial \mu_k^{-1}}(I-\widetilde{\Omega})^{-1} = +\beta'(I-\widetilde{\Omega})^{-1}\frac{\partial S}{\partial \mu_k^{-1}}\Omega(I-\widetilde{\Omega})^{-1} = \beta'(I-\widetilde{\Omega})^{-1}\frac{\partial S}{\partial \mu_k^{-1}}S^{-1}\widetilde{\Omega}(I-\widetilde{\Omega})^{-1}.$$

Note that $\frac{\partial S}{\partial \mu_k^{-1}}S^{-1} = \mu_k v_k v_k'$ with $v_k v_k'$ is the $(N\times N)$ matrix such that all elements are zeros except the k^{th} of the diagonal. Note also that $\widetilde{\Omega}(I-\widetilde{\Omega})^{-1} = (I-\widetilde{\Omega})^{-1} - I = \widetilde{\Omega} + \widetilde{\Omega}^2 + \dots$ Using this expression in the above equation yields:

$$\frac{\partial \widetilde{\beta}'}{\partial (\mu_k^{-1})} = \mu_k \beta'(I-\widetilde{\Omega})^{-1}v_k v_k'\left[(I-\widetilde{\Omega})^{-1} - I\right] = \mu_k \widetilde{\beta}_k v_k'\left[(I-\widetilde{\Omega})^{-1} - I\right].$$

□

Back to the proof of Proposition 10, let us recall that $\log Y^u = -\log(1-\frac{Pro}{P^C C})$. By using the chain rule, we have

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = \frac{\partial \log Y^u}{\partial \log \overline{Z_k}}\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} + \frac{\partial \log Y^u}{\partial \log \Delta_k}\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{\partial \log Y^u}{\partial \log \Delta_k}\frac{\partial \log \Delta_k}{\partial \log Z(k,i)},$$

where the last equality comes from the fact that $\frac{\partial \log Y^u}{\partial \log \overline{Z_k}} = 0$. Indeed, $\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\{1-\mu_k^{-1}\}_k$ is entirely determined by the parameters β and Ω and the markups μ_k, and the latter are themselves entirely determined by the Δ_k. Using the expression of $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)}$, we have

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = \frac{2}{\Delta_k}\left(\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} - \Delta_k\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}\frac{\partial \log Y^u}{\partial \log \Delta_k}. \tag{18}$$

Let us compute $\frac{\partial \log Y^u}{\partial \log \Delta_k}$:

$$\frac{\partial \log Y^u}{\partial \log \Delta_k} = \frac{\Delta_k}{1-\frac{Pro}{P^C C}}\frac{\partial \frac{Pro}{P^C C}}{\partial \Delta_k} = \frac{\Delta_k}{\frac{wL}{P^C C}}\frac{\partial(\mu_k^{-1})}{\partial \Delta_k}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})} = -\frac{f_k(\Delta_k)}{\varepsilon_k}\frac{\Delta_k f_k'(\Delta_k)}{f_k(\Delta_k)}\frac{P^C C}{wL}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})},$$

where I use the chain rule in the second equality and the expression of the sector k's markup $\mu_k^{-1} = 1 - \frac{f_k(\Delta_k)}{\varepsilon_k}$ in the last equality. Note that I can use the chain rule in the second equality because the markup in sector k is entirely determined by the index Δ_k. Using the definition of the markup and the elasticity e_k of the function f_k, we have

$$\frac{\partial \log Y^u}{\partial \log \Delta_k} = -(1-\mu_k^{-1})e_k\frac{P^C C}{wL}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})}. \tag{19}$$

Let us compute $\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})}$. First thanks to Proposition 8, we have $\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\{1-\mu_k^{-1}\}_k = \widetilde{\beta}'\{1-\mu_k^{-1}\}_k$ with the notation of Lemma 3: $\widetilde{\beta} = \left\{\frac{P_k Y_k}{P^C C}\right\}_k = \beta'(I-\widetilde{\Omega})^{-1}$. Using the fact that for two vectors y and x that are function of z then $\frac{\partial(y'x)}{\partial z} = x'\frac{\partial y}{\partial z} + y'\frac{\partial x}{\partial z}$, we have

$$\begin{aligned}
\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})} &= \widetilde{\beta}'\frac{\partial(\{1-\mu_k^{-1}\}_k)}{\partial(\mu_k^{-1})} + \{1-\mu_k^{-1}\}_k'\frac{\partial \widetilde{\beta}}{\partial(\mu_k^{-1})} \\
&= -\widetilde{\beta}'v_k + \mu_k\widetilde{\beta}_k\{1-\mu_k^{-1}\}_k'\left[\left((I-\widetilde{\Omega})^{-1}\right)' - I\right]v_k \\
&= -\widetilde{\beta}_k + \mu_k\widetilde{\beta}_k\left[\left(\{1-\widetilde{\mu_k}^{-1}\}_k\right)'v_k - \{1-\mu_k^{-1}\}_k'v_k\right] \\
&= -\widetilde{\beta}_k + \mu_k\widetilde{\beta}_k\left[1-\widetilde{\mu_k}^{-1} - 1 + \mu_k^{-1}\right] = -\widetilde{\beta}_k\frac{\mu_k}{\widetilde{\mu_k}} = -\frac{P_k Y_k}{P^C C}\frac{\mu_k}{\widetilde{\mu_k}}
\end{aligned} \tag{20}$$

where in the fifth line I use the definition of $\widetilde{\mu_k}^{-1}$ in Proposition 8. Substituting Equations 20 and 19 into

Equation 18 yields the result:

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = -\frac{Pro_k}{wL}\frac{\mu_k}{\widetilde{\mu_k}}\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}.$$

where I use the fact that $Pro_k = P_k Y_k(1-\mu_k^{-1})$. $\square$

A.6 Proof of Proposition 11 (Elasticity of Sector-Level Price)

Using Equation 2 and the results of Proposition 5, we have

$$\{\log P_k\}_k = (I-\Omega)^{-1}\left\{\log \overline{Z_l}^{\frac{-1}{\varepsilon_l-1}} f_l(\Delta_l)^{\frac{1}{\varepsilon_l-1}}\frac{\varepsilon_l}{\varepsilon_l-1}\right\}_l + \log w\mathbb{I},$$

where I used the fact that $(I-\Omega)^{-1}\{\gamma_l\}_l = \mathbb{I}$, where $\mathbb{I}=\{1\}_l$. Taking the derivative with respect to $Z(l,i)$ yields

$$\left\{\frac{\partial \log P_k}{\partial \log Z(l,i)}\right\}_k = \frac{(I-\Omega)^{-1}v_l}{\varepsilon_l-1}\left(-\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,i)} + \frac{\partial \log \Delta_l}{\partial \log Z(l,i)}e_l\right) + \log w\mathbb{I},$$

where e_l is the elasticity of f_l and v_l is the $N\times 1$ vector where the element l is 1 and the others are zero. Note that $\psi^d v_l = (I-\Omega)^{-1}v_l = \{\psi^d_{k,l}\}_k$. Let us substitute the expression of $\frac{\partial \log \Delta_l}{\partial \log Z(l,i)}$ to find that

$$\left\{\frac{\partial \log P_k}{\partial \log Z(l,i)}\right\}_k = -\frac{\{\psi^d_{k,l}\}_k}{\varepsilon_l-1}\left(1+\frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,i)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\right)\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,i)} + \log w\mathbb{I}.$$

$\square$

A.7 Proof of Proposition 12 (Elasticity of Sector-Level Sales Share)

From Equation 3 of Proposition 5, the sales share of sectors are such that

$$\left\{\frac{P_kY_k}{P^CC}\right\}_k' = \beta'(I-\widetilde{\Omega})^{-1},$$

Using the chain rule,

$$\left\{\frac{\partial \log\left(\frac{P_kY_k}{P^CC}\right)}{\partial \log Z(l,i)}\right\}_k' = \left\{\frac{\mu_l^{-1}}{\frac{P_kY_k}{P^CC}}\frac{\partial \log(\mu_l^{-1})}{\partial \log Z(l,i)}\frac{\partial\left(\frac{P_kY_k}{P^CC}\right)}{\partial(\mu_l^{-1})}\right\}_k',$$

Lemma 3 gives that $\left\{\frac{\partial\left(\frac{P_kY_k}{P^CC}\right)}{\partial(\mu_l^{-1})}\right\}_k' = \mu_l\widetilde{\beta}_l v_l'\left[(I-\widetilde{\Omega})^{-1}-I\right]$. Because $v_l'\psi^s = \{\psi^s_{l,k}\}_k$ we have $\frac{\partial\left(\frac{P_kY_k}{P^CC}\right)}{\partial(\mu_l^{-1})} = \mu_l\widetilde{\beta}_l(\psi^s_{l,k}-\mathbb{I}_{l,k})$, where $\mathbb{I}_{l,k}=1$ is $l=k$ and $=0$ otherwise. Using the expression of $\frac{\partial\log(\mu_l^{-1})}{\partial \log Z(l,i)}$, the fact that $\widetilde{\beta}_l = \frac{P_lY_l}{P^CC}$, we have

$$\left\{\frac{\partial \log\left(\frac{P_kY_k}{P^CC}\right)}{\partial \log Z(l,i)}\right\}_k' = \left\{\frac{P_lY_l}{P_kY_k}\frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,i)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)(\psi^s_{l,k}-\mathbb{I}_{l,k})\right\}_k',$$

$\square$

B Data Appendix

In this paper, I use two types of data at the sector level. The first type is the oinput-output I-O data of the Bureau of Economic Analysis (BEA). The second type is the concentration data of the US Census Bureau.
The BEA provide I-O information at different level of aggregation. I use here the detailed I-O table from 2007, which gives information on 389 sectors. The BEA does not provide direct requirement Industry-by-Industry table but instead total Industry-by-Industry requirement table, TOT. I then use the formula

$\widetilde{\Omega} = (TOT - I)TOT^{-1}$ to find the direct requirement of an industry input to produce one dollar of its output at the steady state. To find the value of household-consumption share, I use the BEA's USE table, which gives for each commodity how much the household buys of this commodity. I then recover the share of income the household spends on each industry, by premultiplying these commodity spending shares by the MAKE table. For each industry, the MAKE table gives how much of each commodity is needed to produce one dollar of output.

The US Census Bureau provides a concentration measure for different levels of aggregation all sectors except for Agriculture, Forestry, Fishing and Hunting (11); Mining, Quarrying, and Oil and Gas Extraction (21); Construction (23); Management of Companies and Enterprises (55); Public Administration (92). The measure of concentration are the top 4,8,20, and 50 firms' share of total industry revenues in 2002, 2007, and 2012. For manufacturing (31-33), the US Census Bureau also gives the Herfindahl-Hirschman Index among the 50 largest firms. I use these data for Figures 9 and 8 in Online Appendix E. The former plots the sector-level concentration measure in 2002 versus 2007. The latter displays the empirical distribution of the sector-level concentration measures.

Using the correspondence table given by the BEA between the I-O sectors-classification and the NAICS 2007 classification, I matched these two data sources to plot Figure 1 and to calibrate the model in section 6.

C Numerical Appendix

In this appendix, I first describe how to simulate a path of productivity for each of the 5.6 million firms in an efficient way. Second, I describe how to numerically solve for the equilibrium allocation without relying on Assumption 2.

Simulation of a Path of Productivity Distributions

To simulate a path of productivity for a large number of firms, I follow the number of firms in each productivity bins rather than the productivity of each firm. The idea is the one described in Proposition 6 and is illustrated in the discussion of the simple example of Figure 5. The key assumption is that productivity evolves on a discrete grid: the number of firms in each bin characterizes the whole distribution of productivity across firms. Because firms in a same productivity bin are the same, following the number of firms in each productivity bin is equivalent to following the productivity of each firm.

The simulation procedure follows closely the proof of Proposition 6 in Appendix F.2. For a given period t, for a given sector k, and for a given distribution of firms $g_t^{(k)}$ in this sector, that is, a vector whose elements are the number of firms in each productivity bin, we know the number of firms in each productivity bin at time $t+1$ that were in productivity bin n at t follow a multinomial random vector with the number of trials being $g_{t,n}^{(k)}$ and the event probability given by the n^{th} row vector of the matrix $\mathcal{P}^{(k)}$. The next productivity distribution $g_{t+1}^{(k)}$ is just the sum of all these conditional distributions. This procedure makes the simulation of a path of productivity for each of the 5.6 million firms extremely efficient. I use this procedure in all the simulation exercises in the main text of this paper.

Solving the Equilibrium Allocation

Given the distributions of productivity across firms in each sector, I can solve for the equilibrium allocation. The first step is to solve for the Bertrand firm-level problem described in Proposition 1. Note that after substituting for the firm's marginal cost, and defining $X_k = \left(\frac{\varepsilon_k}{\varepsilon_k - 1} P_k^{-1} w^{\gamma_k} \prod_{l=1}^N P_l^{\omega_{k,l}}\right)^{1-\varepsilon_k}$, this problem is equivalent to

$$\forall i \in [|1, N_k|], \begin{cases} s(k,i) & = \left(\mu(k,i) Z(k,i)^{-\gamma_k} \frac{\varepsilon_k - 1}{\varepsilon_k}\right)^{1-\varepsilon_k} X_k \\ \mu(k,i) & = \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} \end{cases} \quad \text{and} \quad X_k = \left(\sum_{i=1}^{N_k} \mu(k,i)^{1-\varepsilon_k} Z(k,i)^{\gamma_k(\varepsilon_k - 1)} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{\varepsilon_k - 1}\right)^{-1}$$

Given firm-level productivities $Z(k,i)$, the above system of equations can be solved numerically and gives, for each sector, the firm-level sales share $s(k,i)$, the markups $\mu(k,i)$, and X_k. Note that one equation exists per firm, so when the number of firms is very large, as it is in the baseline calibration, the size of this system becomes too large. To save computation time, I rewrite the above system for each possible productivity level and use the number of firms in each productivity bin given by the vector $g_t^{(k)}$: the sum in the right-hand side of the above equation satisfied by X_k is now over the productivity bins rather than the firms:

$$\forall n \in [|0, M_k|], \begin{cases} s(k,n) & = \left(\mu(k,n) (\varphi_k^n)^{-\gamma_k} \frac{\varepsilon_k - 1}{\varepsilon_k}\right)^{1-\varepsilon_k} X_k \\ \mu(k,n) & = \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,n)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,n)} \end{cases} \quad \text{and} \quad X_k = \left(\sum_{n=0}^{M_k} \mu(k,n)^{1-\varepsilon_k} (\varphi_k^n)^{\gamma_k(\varepsilon_k - 1)} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{\varepsilon_k - 1} g_{t,n}^{(k)}\right)^{-1}$$

where $\mu(k,n)$ and $s(k,n)$ stand for the markup and the sales share of firms with productivity level φ_k^n. The number of equations in this system is the number of productivity bins M_k in sector k and is independent of the number of firms N_k in sector k.
With the distribution of sales share and markups across firms in each sector, I can compute sector-level productivities and markups as they are defined in Section 3.2. Given these sector-level markups and productivities, the equilibrium allocation is entirely characterized by Proposition 8.

D Elastic Labor Supply

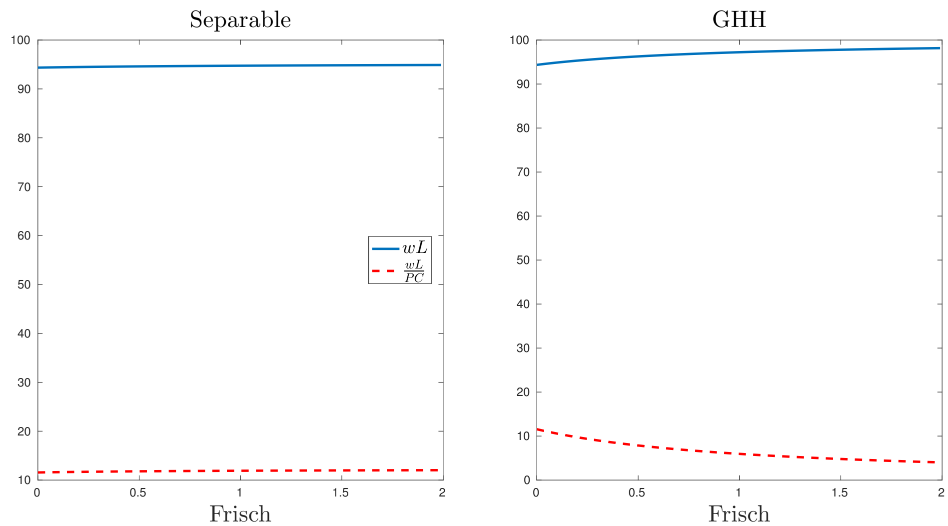
In this appendix, I show how the main results are affected by relaxing the inelastic-labor-supply assumption. I consider the case of separable and GHH preferences. In both cases, aggregate output Y_t is a function of the equilibrium wage and the profit share as in the inelastic cases (Equation 7). With separable preferences $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta\frac{L^{1+1/f}}{1+1/f}$, where f is the Frisch elasticity of the labor supply, η is the coefficient of relative risk aversion, and aggregate output is $\log Y_t = \left(1 + \frac{1-\eta}{1/f+\eta}\right)\log w_t - \left(1 - \frac{\eta}{1/f+\eta}\right)\log\left(1 - Pro_t/(P_t^C C_t)\right)$. With GHH preferences $U(C,L) = \frac{1}{1-\eta}\left(C - \theta\frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$, aggregate output is $\log Y_t = (1+f)\log w_t - \log\left(1 - Pro_t/(P_t^C C_t)\right)$.
Let us define the "downstream" and "upstream" parts of aggregate output as in Section 4, that is, the (log) labor income and the (log) inverse of the labor income share, respectively. With GHH preferences, the elasticity in Proposition 9 is multiplied by $(1+f)$, whereas the result in Proposition 10 is unaffected. With these preferences, the income effect does not affect the labor supply, and labor income is only a function of the equilibrium wage: $w_t L_t = w_t^{1+f}\theta^{-f}$. With separable preferences, labor income is a function of aggregate output, $w_t L_t = w_t^{1+f}\theta^{-f}C_t^{-\eta f}$, and the elasticity of the "downstream" part is a weighted sum of the elasticities in Propositions 9 and 10. The elasticity of the "upstream" part is unaffected.
The relative standard deviation of the "downstream" and "upstream" parts of aggregate output with respect to aggregate volatility for different preferences can be found in Table 6. The first row is for separable preferences, whereas the second is for GHH preference. In every row, the calibration is as in the baseline case of Table 2 with $\eta = 1$ and a Frisch elasticity of $f = 2$. Figure 7 plots these relative standard deviations for a Frisch elasticity varying from 0 to 2.

Table 6: Aggregate Volatility and Elastic Labor Supply

	Downstream	Upstream
Inelastic	93.55	11.29
Separable	94.90	12.02
GHH	98.16	4.01

NOTE: Each row is the relative standard deviation with respect to (log) aggregate output of the (log) labor income and (log) labor share, that is, the "downstream" and "upstream" parts of aggregate output. The first row is the baseline case of inelastic labor supply (as in Table 3). The second row is the case of separable preference: $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta\frac{L^{1+1/f}}{1+1/f}$. The third row is the case of GHH preferences: $U(C,L) = \frac{1}{1-\eta}\left(C - \theta\frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$. The calibration is as in Table 2 with $\eta = 1$ and $f = 2$. Numbers are reported in percentage points. These statistics come from $4,000$-period simulations.

Figure 7: Aggregate Volatility and Frisch



NOTE: Relative standard deviation with respect to the (log) aggregate output of the (log) labor income and the (log) labor share as a function of the Frisch elasticity of labor supply. Left panel: for separable preferences, $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta\frac{L^{1+1/f}}{1+1/f}$. Right panel: for GHH preferences, $U(C,L) = \frac{1}{1-\eta}\left(C - \theta\frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$. The calibration is as in Table 2 with $\eta = 1$ and $f = 2$. Numbers are reported in percentage points. These statistics come from $4,000$-period simulations.

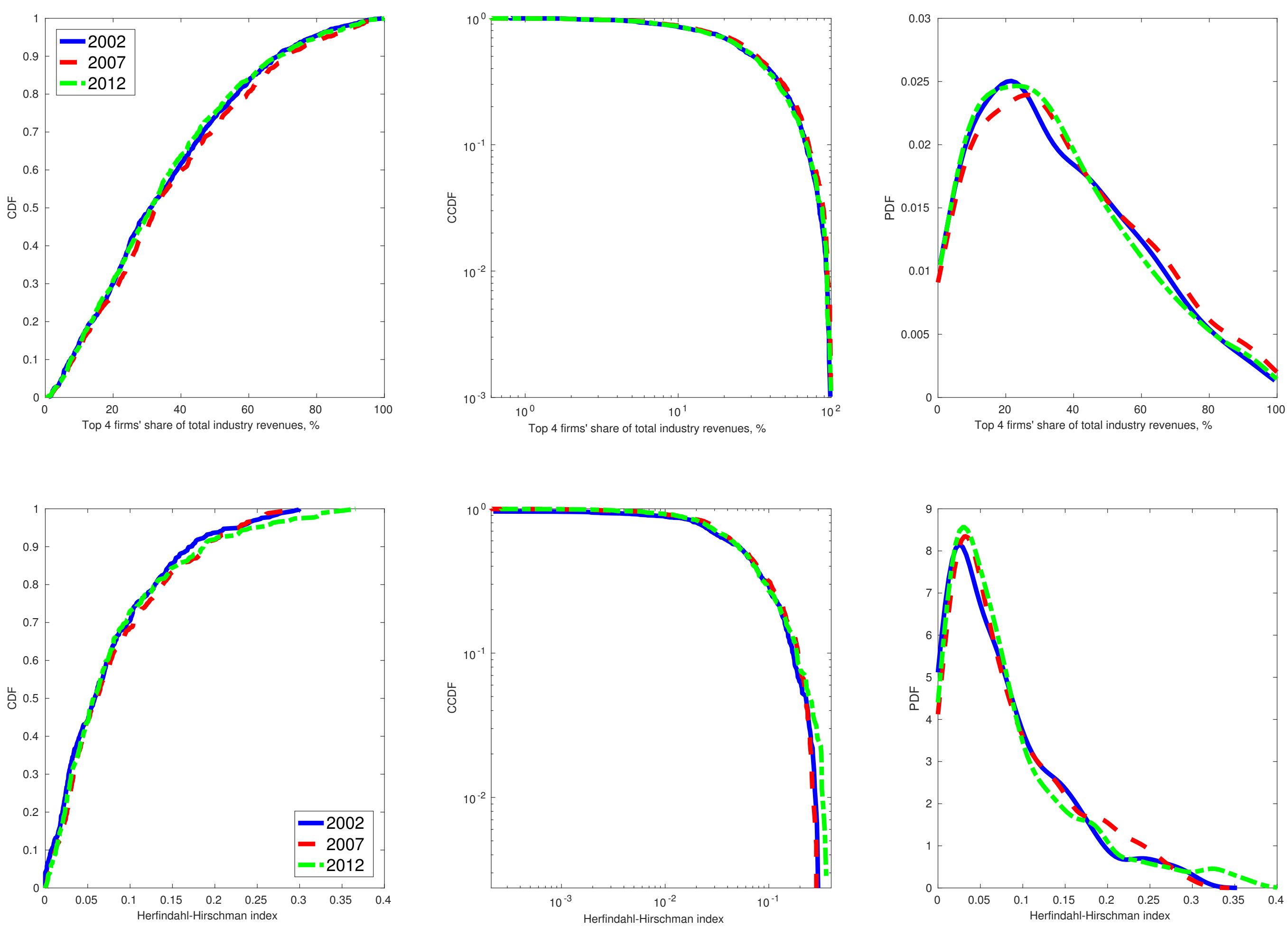
Online Appendix to
“IO in I-O: Size, Industrial Organization and the Input-Output Network Make a Firm Structurally Important”

Basile Grassi

Not for Publication. January 26, 2018

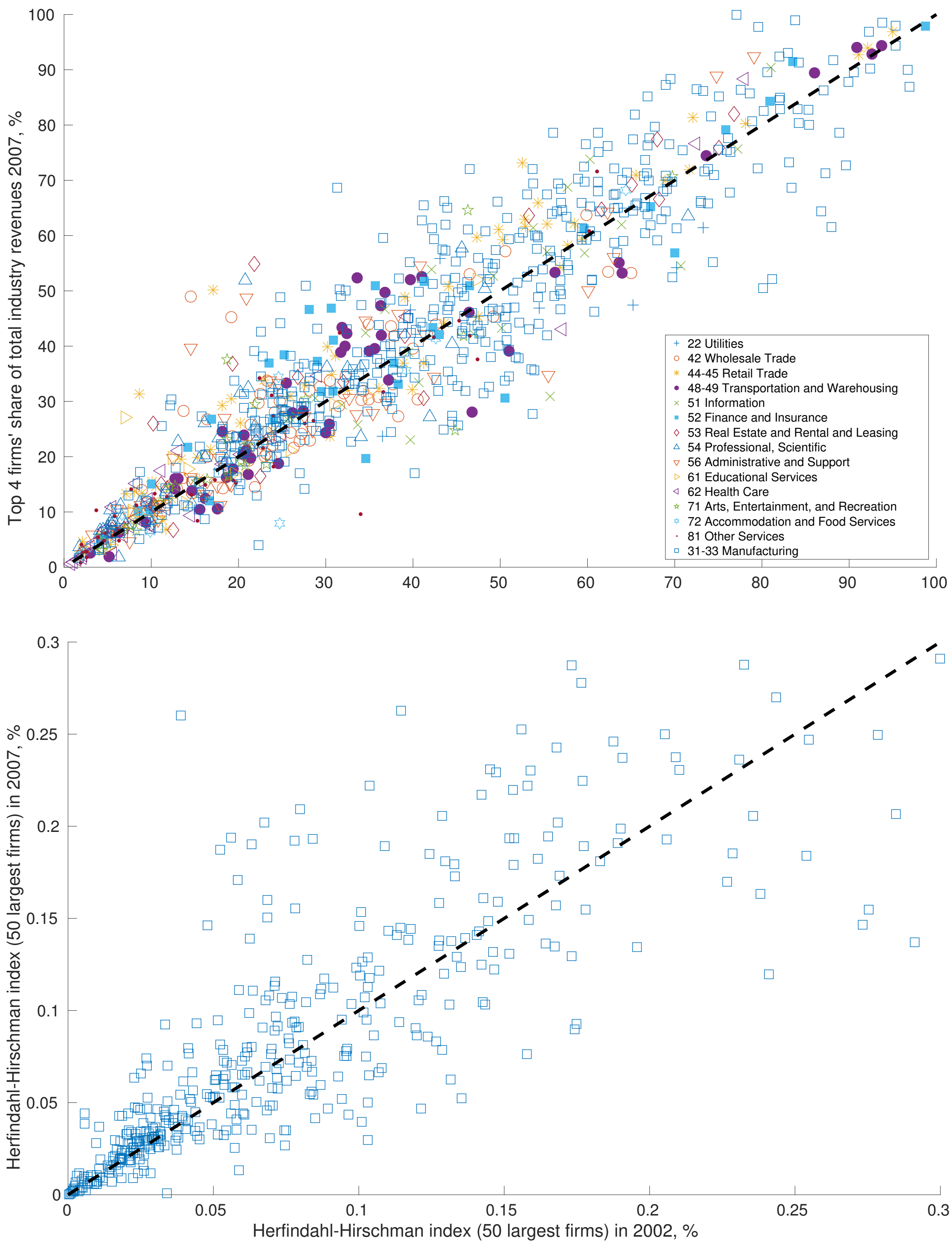
E Figures Appendix

Figure 8: Sectors’ Concentration Distribution



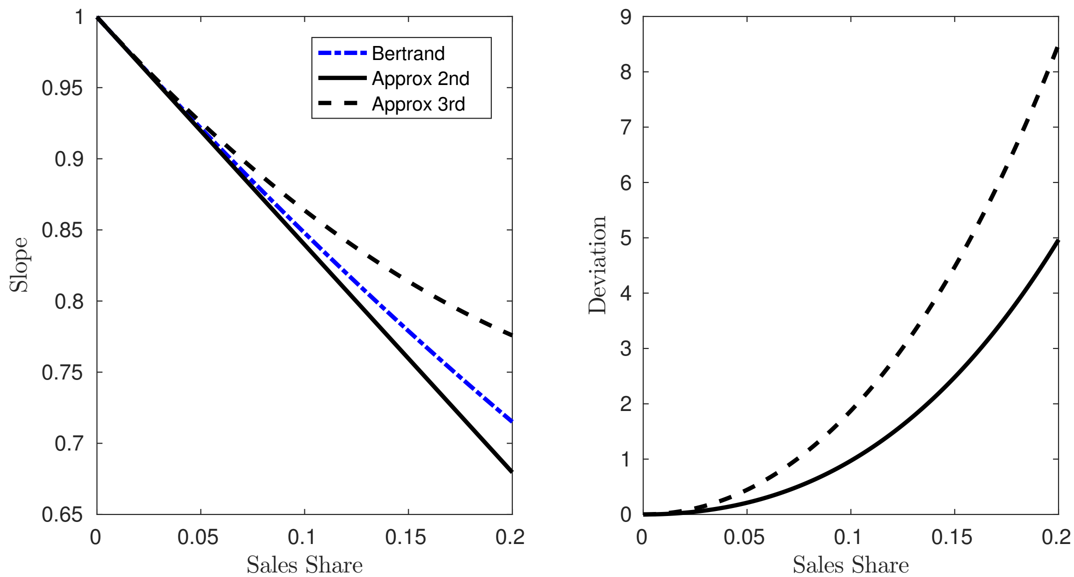
NOTE: Empirical cumulative distribution function (left), counter cumulative distribution function (center), and kernel-smoothing-function estimate of the probability distribution function (right) of the top four firms’ share of total revenues for 6-digit NAICS industries (top panel) and of the Herfindahl-Hirschman Index for the 50 largest companies for the 6-digit NAICS manufacturing industries (31-33). Data: US Census Bureau.

Figure 9: Sector Concentration



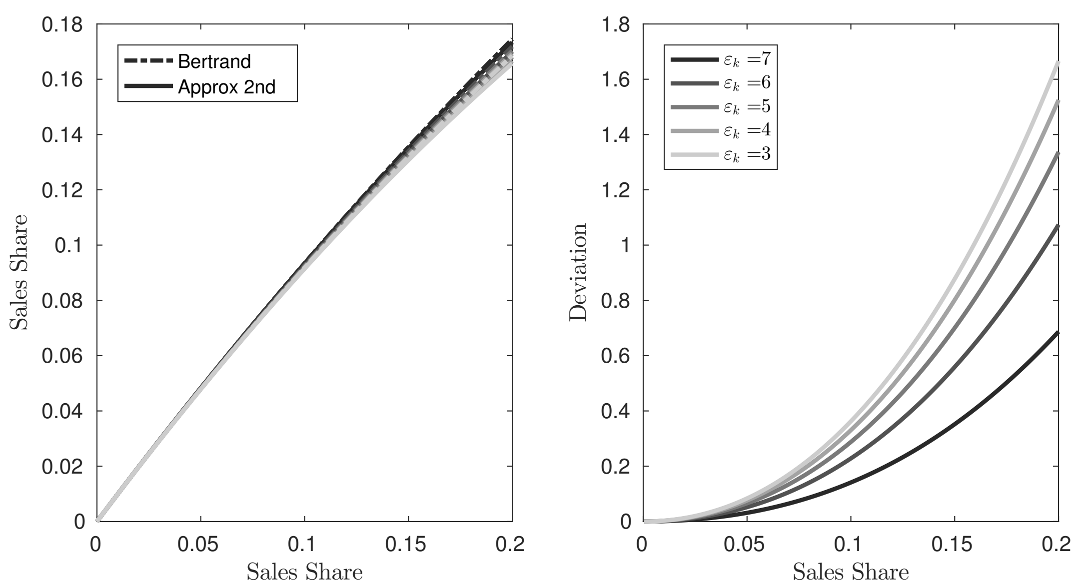
NOTE: Top Panel: Top four firms' share of total revenues in 2002 and 2007 for 6-digit NAICS industries. The mean value is 35.37% in 2002 and 37.21% in 2007. 970 industries. Bottom Panel: Herfindahl-Hirschman-Index for the 50 largest companies in 2002 and in 2007 for 6-digit NAICS manufacturing industries (31-33). 448 industries. Data: US Census Bureau.

Figure 10: Approximation of Firms' Sales Share (Slope)



NOTE: For $\varepsilon_k = 5$. The left panel shows the slope of the Bertrand sales share, and the slopes of the second- and the third-order approximations in Proposition 2 as a function of the monopolistic sales share. The right panel shows the percentage deviation of the slope of both approximations with respect to the numerical solution.

Figure 11: Approximation of Firms' Sales Share (Different ε_k)



NOTE: For different values of ε_k. The left panel shows the Bertrand sales share using a numerical solver (solid) and the second-order approximation (dashed) as a function of the monopolistic sales share. The right panel shows the percentage deviation of the second-order approximation with respect to the numerical solution.

F Proof Appendix

F.1 Proof of Proposition 3 (Size-Volatility)

Let us first compute the variance of the growth rate of productivity. Let us call $n_{t,k,i}$ the integer such that the productivity level of firm i in sector k is such that $Z_t(k,i) = \varphi_k^{n_{t,k,i}}$. Note that $Z_t(k,i)$ follows the Markovian process described in Assumption 1; therefore, its growth rate satisfies

$$\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} = \frac{\varphi_k^{n_{t+1,k,i}} - \varphi_k^{n_{t,k,i}}}{\varphi_k^{n_{t,k,i}}} = \varphi_k^{n_{t+1,k,i} - n_{t,k,i}} - 1 = \begin{cases} \varphi_k^{-1} - 1 & & a \\ 0 & \text{with proba} & b \\ \varphi_k - 1 & & c \end{cases}$$

Let us compute the conditional expected growth rate of the productivity of firm i in sector k:

$$\mathbb{E}_t \left[\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} \right] = a\left(\varphi_k^{-1} - 1\right) + c\left(\varphi_k - 1\right) = a\varphi_k^{-1} + b + c\varphi_k - 1,$$

whereas the conditional variance of the growth rate of $Z_t(k,i)$ is

$$\mathbb{V}ar_t \left[\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} \right] = a\left(\varphi_k^{-1} - 1\right)^2 + c\left(\varphi_k - 1\right)^2 - (a\varphi_k^{-1} + b + c\varphi_k - 1)^2 = \sigma_k^2,$$

These conditional moments are independent of the level $Z_t(k,i)$ at time t and they are equal to their unconditional counterpart. This completes the first part of the proof.
Let us now turn to the growth rate of the sales share $s_t(k,i)$ of firm i in sector k. To this end, I use the approximation in Assumption 2. The first step is to find the growth rate of the sales share under monopolistic competition $\hat{s}_t(k,i)$. Note that $\hat{s}_t(k,i) \propto Z_t(k,i)^{-\gamma_k(1-\varepsilon_k)}$. Keeping sectors' price and the wage constant, at the first-order, we have $g_{t+1}^{\hat{s}(k,i)} = -\gamma_k(1-\varepsilon_k) g_{t+1}^{Z(k,i)}$, where $g_{t+1}^{x} = \frac{x_{t+1} - x_t}{x_t}$.[33] Let us focus on the case of Bertrand competition. All the following calculation, are very similar under Cournot. Thanks to Assumption 2, the sales share of firm i in sector k is such that $s_t(k,i) = \hat{s}_t(k,i) - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)^2$, which becomes

$$g_{t+1}^{s(k,i)} = \frac{\hat{s}_t(k,i)}{s_t(k,i)} g_{t+1}^{\hat{s}(k,i)} - 2\frac{(1-\varepsilon_k^{-1})\hat{s}_t(k,i)^2}{s_t(k,i)} g_{t+1}^{\hat{s}(k,i)}$$

Using the fact that $g_{t+1}^{\hat{s}(k,i)} = -\gamma_k(1-\varepsilon_k) g_{t+1}^{Z(k,i)}$ and after some simplification, we have

$$g_{t+1}^{s(k,i)} = \gamma_k(\varepsilon_k - 1) \frac{1 - 2(1-\varepsilon_k^{-1})\hat{s}_t(k,i)}{1 - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)} g_{t+1}^{Z(k,i)}$$

The conditional variance of the growth rate of firm i in sector k is

$$\mathbb{V}ar_t \left[\frac{s_{t+1}(k,i) - s_t(k,i)}{s_t(k,i)} \right] = \gamma_k^2(\varepsilon_k - 1)^2 \left(\frac{1 - 2(1-\varepsilon_k^{-1})\hat{s}_t(k,i)}{1 - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)} \right)^2 \sigma_k^2.$$

The above equation shows the variance of the growth rate of the sales share of a firm is a strictly decreasing function of its level. Indeed, the function $g_k : x \mapsto \gamma_k^2(\varepsilon_k - 1)^2 \frac{1-2(1-\varepsilon_k^{-1})x}{1-(1-\varepsilon_k^{-1})x}$ is strictly decreasing and the absolute value of its slope $|g_k'(x)| = \gamma_k^2(\varepsilon_k - 1)^2 \frac{(1-\varepsilon_k^{-1})}{(1-(1-\varepsilon_k^{-1})x)^2}$ is strictly increasing in ε_k. □

F.2 Proof of Proposition 6 (Sector k's Productivity Distribution Dynamics)

In this section, I first derive equation 4 before solving for the stationary distribution in sector k.

[33] Equivalently, one can compute this growth rate under the stationary equilibrium, the steady-state of this economy where aggregate and sectoral quantities and prices are constant (as if they were a continuum of sectors).

Proof of Equation 4: For n such that $0 < n < M_k$, Assumption 1 implies

$$g_{t+1,n}^{(k)} = f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1},$$

where $f_{k,t+1}^{n',n}$ is the number of firms in productivity bin n' at $t+1$ that were in bin n at time t. Thanks to Assumption 1, the 3×1 vector $f_{k,t+1}^{\cdot,n} = (f_{k,t+1}^{n-1,n}, f_{k,t+1}^{n,n}, f_{k,t+1}^{n+1,n})'$ follows a multinomial distribution with the number of trial, $g_{t,n}^{(k)}$, and event probabilities $(a_k, b_k, c_k)'$. It follows that the 3×1 vector $f_{k,t+1}^{\cdot,n}$ has a mean $\mathbb{E}_t\left[f_{k,t+1}^{\cdot,n}\right] = g_{t,n}^{(k)}(a_k, b_k, c_k)'$ and a variance-covariance matrix equal to $g_{t,n}^{(k)}\Sigma_k$, where

$$\Sigma_k = \begin{pmatrix} a_k(1-a_k) & -a_k b_k & -a_k c_k \\ -a_k b_k & b_k(1-b_k) & -b_k c_k \\ -a_k c_k & -b_k c_k & c_k(1-c_k) \end{pmatrix}.$$

Note that $f_{k,t+1}^{\cdot,n}$ are independent across n, and thus

$$\mathbb{E}_t\left[g_{t+1,n}^{(k)}\right] = \mathbb{E}_t\left[f_{k,t+1}^{n,n-1}\right] + \mathbb{E}_t\left[f_{k,t+1}^{n,n}\right] + \mathbb{E}_t\left[f_{k,t+1}^{n,n+1}\right] = a_k g_{t,n+1}^{(k)} + b_k g_{t,n}^{(k)} + c_k g_{t,n-1}^{(k)}$$

$$\mathbb{V}ar_t\left[g_{t+1,n}^{(k)}\right] = \mathbb{V}ar_t\left[f_{k,t+1}^{n,n-1}\right] + \mathbb{V}ar_t\left[f_{k,t+1}^{n,n}\right] + \mathbb{V}ar_t\left[f_{k,t+1}^{n,n+1}\right] = a_k(1-a_k)g_{t,n+1}^{(k)} + b_k(1-b_k)g_{t,n}^{(k)} + c_k(1-c_k)g_{t,n-1}^{(k)}.$$

For completeness, let us look at the covariance structure of the $g_{t+1,n}^{(k)}$:

$$\mathbb{C}ov_t\left[g_{t+1,n}^{(k)}; g_{t+1,n'}^{(k)}\right] = \mathbb{C}ov_t\left[f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n',n'-1} + f_{k,t+1}^{n',n'} + f_{k,t+1}^{n',n'+1}\right] = 0 \text{ if } |n-n'| > 2,$$

because the $f_{k,t+1}^{\cdot,n}$ are independent across n. For $n' = n+1$, we have

$$\begin{aligned} \mathbb{C}ov_t\left[g_{t+1,n}^{(k)}; g_{t+1,n+1}^{(k)}\right] &= \mathbb{C}ov_t\left[f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n+1,n} + f_{k,t+1}^{n+1,n+1} + f_{k,t+1}^{n+1,n+2}\right] \\ &= \mathbb{C}ov_t\left[f_{k,t+1}^{n,n}; f_{k,t+1}^{n+1,n}\right] + \mathbb{C}ov_t\left[f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n+1,n+1}\right] \\ &= -b_k c_k g_{t,n}^{(k)} - a_k b_k g_{t,n+1}^{(k)} \end{aligned}$$

using the fact the variance-covariance matrix of $f_{k,t+1}^{\cdot,n}$ is equal to $\mu_{t,n}^{(k)}\Sigma_k$ for all $n > 0$. The same reasoning apply for $n' = n+2$.
For $n = 0$, Assumption 1 implies

$$g_{t+1,0}^{(k)} = f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1}$$

and that the 2×1 vector $f_{k,t+1}^{\cdot,0} = (f_{k,t+1}^{0,0}, f_{k,t+1}^{1,0})'$ follows a multinomial distribution with the number of trial $g_{t,0}^{(k)}$ and event probabilities $(a_k + b_k, c_k)'$. The same reasoning then applies than for $n > 0$.
For $n = M_k$, Assumption 1 implies

$$g_{t+1,M}^{(k)} = f_{k,t+1}^{M,M-1} + f_{k,t+1}^{M,M}$$

and that the 2×1 vector $f_{k,t+1}^{\cdot,M} = (f_{k,t+1}^{M-1,M}, f_{k,t+1}^{M,M})'$ follows a multinomial distribution with the number of trial $g_{t,M}^{(k)}$ and event probabilities $(a_k, c_k + b_k)'$. The same reasoning then applies than for $n > 0$.
Gathering the above results, we have in matrix form,

$$g_{t+1}^{(k)} = (\mathcal{P}^{(k)})' g_t^{(k)} + \epsilon_t^{(k)}$$

where $\epsilon_t^{(k)}$ is the $M \times 1$ vector of $\epsilon_{t,n}^{(k)}$. This completes the derivation of Equation 4.□

Stationary Distribution in Sector k: Let us drop the (k) superscript and subscript to simplify notation. The stationary distribution is a sequence that solves the following system:

$$\begin{aligned} (BC1) \qquad & g_0 = (a+b)g_0 + a g_1 \\ (BC2) \qquad & g_M = c g_{M-1} + (b+c) g_M \\ (EH) \qquad & g_n = a g_{n+1} + b g_n + c g_{n-1}. \end{aligned}$$

Let us solve for the general solution of (EH). This equation is a second-order linear-difference equation equiv-

alent to $0 = ag_{n+1} + (b-1)g_n + cg_{n-1} = ag_{n+1} - (a+c)g_n + cg_{n-1}$, with an associated second-order polynomial $aX^2 - (a+c)X + c = 0$ that has roots 1 and $\frac{c}{a}$. The general solution of (EH) is thus $g_n = K_1 + K_2\left(\frac{c}{a}\right)^n$, where K_1 and K_2 are constant to solve for.
Let us substitute this general solution in the equation $(BC1)$. Doing so yields

$$K_1 + K_2 = (a+b)(K_1+K_2) + aK_1 + aK_2\frac{c}{a} = (2a+b)K1 + (a+b+c)K2$$

because $a+b+c=1$, $(BC1)$ implies $K_1 = (2a+b)K_1$. Because $a<c$ and $a+b+c=1$, $2a+b \neq 1$ and thus $K_1 = 0$. The general solution of this system is then $g_n = K_2\left(\frac{c}{a}\right)^n$. It is trivial to see that $(BC2)$ is satisfied by this general solution. Because $n = \frac{\log\varphi^n}{\log\varphi}$, $\left(\frac{c}{a}\right)^n = \exp\left(-s\log\frac{a}{c}\right) = \exp\left(-\frac{\log\varphi^n}{\log\varphi}\log\frac{a}{c}\right) = (\varphi^n)^{-\delta}$ with $\delta = \frac{\log\frac{a}{c}}{\log\varphi}$.
It follows that $g_n = K_2\left(\varphi^n\right)^{-\delta}$
To solve for K_2, let us use the fact that g_n has to sum to N_k, the number of firms in sector k:

$$N_k = \sum_{n=0}^{M} g_n = K_2\sum_{n=0}^{M}\left(\varphi^{-\delta}\right)^n = K_2\frac{1-\left(\varphi^{-\delta}\right)^{M+1}}{1-\varphi^{-\delta}},$$

because $\varphi^{-\delta} < 1$. It follows that $K_2 = N_k\frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}}$ and $g_n^{(k)} = N_k\frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}}(\varphi^n)^{-\delta}$. □

F.3 Proof of Proposition 7 (Dynamics of $\overline{Z_{t,k}}$ and $\Delta_{t,k}$)

Let us define $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi$, the ξth moment of the productivity distribution within sector k at time t. Note that because productivity evolves on the discrete state space $\Phi_k = \{1, \varphi_k, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\}$, we can rewrite $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi = \sum_{i=1}^{N_k}\varphi_k^{\xi n_{t,k,i}}$, where $n_{t,k,i}$ is such that firm i in sector k has a productivity level $\varphi^{n_{t,k,i}}$ at time t. It follows that $MZ_{t,k}(\xi) = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t,n}^{(k)}$ by, instead of summing over firms i, summing over productivity level φ_k^n. Below, I show two lemmas that describe the dynamics of the moments $MZ_{t,k}(\xi)$ for any ξ. With these results in hand, I characterize the dynamics of the two moments of interest: $\overline{Z_{t,k}}$ and $\Delta_{t,k}$.

Lemma 4 ***(Dynamics of Moments of the Productivity Distribution)*** *Under Assumption 1, the ξth moment of the productivity distribution within sector k, $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^\xi$, satisfies*

$$MZ_{t+1,k}(\xi) = \rho_k(\xi)MZ_{t,k}(\xi) + O_{t,k}^M(\xi) + \sigma_{t,k}(\xi)\varepsilon_t$$
$$\sigma_{t,k}(\xi)^2 = \varrho_k(\xi)MZ_{t,k}(2\xi) + O_{t,k}^\sigma(\xi),$$

where ε_t is an iid (across t and k) random variable following a $\mathcal{N}(0,1)$, where $\rho_k(\xi) = a_k\varphi_k^{-\xi} + b_k + c_k\varphi_k^\xi$, and where $\varrho_k(\xi) = a_k\varphi_k^{-2\xi} + b_k + c_k\varphi_k^{2\xi} - \rho_k(\xi)^2$.

Proof of Lemma 4: Note first that

$$MZ_{t+1,k}(\xi) = \sum_{i=1}^{N_k} Z_{t+1}(k,i)^\xi = \sum_{i=1}^{N_k}\varphi_k^{\xi n_{t+1,k,i}} = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t+1,n}^{(k)}$$

where $g_{t+1,n}^{(k)}$ is a stochastic as shown in Proposition 6. In the proof of this proposition, we show that for n such that $0 < n < M_k$,

$$g_{t+1,n}^{(k)} = f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1},$$

where $f_{k,t+1}^{n',n}$ is the number of firms in productivity bin n' at $t+1$ that were in bin n at time t. Given Assumption 1, the 3×1 vector $f_{k,t+1}^{\cdot,n} = (f_{k,t+1}^{n-1,n}, f_{k,t+1}^{n,n}, f_{k,t+1}^{n+1,n})'$ follows a multinomial distribution with the number of trial $g_{t,n}^{(k)}$ and event probabilities $(a_k, b_k, c_k)'$. In other words,

$$f_{k,t+1}^{\cdot,n} = \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{t,n}^{(k)}, \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}\right).$$

Severini (2005) (p377 example 12.7) shows that a multinomial distribution can be approximated (i.e., converge in distribution) by a multivariate normal distribution:

$$\frac{1}{\sqrt{g_{t,n}^{(k)}}}\left(f_{k,t+1}^{\cdot,n}-g_{t,n}^{(k)}\left(\begin{smallmatrix}a_k\\ b_k\\ c_k\end{smallmatrix}\right)\right)\underset{g_{t,n}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z\rightsquigarrow\mathcal{N}(0,\Sigma_k),$$

where $\Sigma_k=\begin{pmatrix}a_k(1-a_k) & -a_kb_k & -a_kc_k\\ -a_kb_k & b_k(1-b_k) & -b_kc_k\\ -a_kc_k & -b_kc_k & c_k(1-c_k)\end{pmatrix}$.

For $n=0$, thanks to Assumption 1, $g_{t+1,0}^{(k)}=f_{k,t+1}^{0,0}+f_{k,t+1}^{0,1}$ and the 2×1 vector $f_{k,t+1}^{\cdot,0}=(f_{k,t+1}^{0,0},f_{k,t+1}^{1,0})'$ follows a multinomial distribution with the number of trial $g_{t,0}^{(k)}$ and event probabilities $(a_k+b_k,c_k)'$. Using the same result in Severini (2005), $\frac{1}{\sqrt{g_{t,0}^{(k)}}}\left(f_{k,t+1}^{\cdot,0}-g_{t,0}^{(k)}\left(\begin{smallmatrix}a_k+b_k\\ c_k\end{smallmatrix}\right)\right)\underset{g_{t,0}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z\rightsquigarrow\mathcal{N}(0,\Sigma_k^{(0)})$, where $\Sigma_k^{(0)}=\begin{pmatrix}c_k(1-c_k) & -c_k(1-c_k)\\ -c_k(1-c_k) & c_k(1-c_k)\end{pmatrix}$.

For $n=M_k$, thanks to Assumption 1, $g_{t+1,0}^{(k)}=f_{k,t+1}^{M,M}+f_{k,t+1}^{M,M-1}$ and the 2×1 vector $f_{k,t+1}^{\cdot,M}=(f_{k,t+1}^{M-1,M},f_{k,t+1}^{M,M})'$ follows a multinomial distribution with the number of trial $g_{t,M}^{(k)}$ and event probabilities $(a_k,b_k+c_k)'$. Using the same result in Severini (2005), $\frac{1}{\sqrt{g_{t,0}^{(k)}}}\left(f_{k,t+1}^{\cdot,M}-g_{t,M}^{(k)}\left(\begin{smallmatrix}a_k\\ b_k+c_k\end{smallmatrix}\right)\right)\underset{g_{t,M}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z\rightsquigarrow\mathcal{N}(0,\Sigma_k^{(M)})$, where $\Sigma_k^{(M)}=\begin{pmatrix}a_k(1-a_k) & -a_k(1-a_k)\\ -a_k(1-a_k) & a_k(1-a_k)\end{pmatrix}$.

Let us keep these results in mind and let us go back to (I drop the subscript k to keep the notation parsimonious)

$$\begin{aligned}
MZ_{t+1,k}(\xi)&=\sum_{n=0}^{M}(\varphi^n)^\xi G_{t+1,n}^{(k)}=g_{t+1,0}^{(k)}+\sum_{n=1}^{M-1}(\varphi^n)^\xi g_{t+1,n}^{(k)}+(\varphi^M)^\xi g_{t+1,M}^{(k)}\\
&=f_{k,t+1}^{0,0}+f_{k,t+1}^{0,1}+\sum_{n=1}^{M-1}(\varphi^n)^\xi\left(f_{k,t+1}^{n,n-1}+f_{k,t+1}^{n,n}+f_{k,t+1}^{n,n+1}\right)+(\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1}+f_{k,t+1}^{M,M}\right)\\
&=f_{k,t+1}^{0,0}+f_{k,t+1}^{0,1}+\sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n-1}+\sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n}+\sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n+1}+(\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1}+f_{k,t+1}^{M,M}\right)\\
&=f_{k,t+1}^{0,0}+f_{k,t+1}^{0,1}+\sum_{n=0}^{M-2}(\varphi^\xi)^{n+1} f_{k,t+1}^{n+1,n}+\sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n}+\sum_{n=2}^{M}(\varphi^\xi)^{n-1} f_{k,t+1}^{n-1,n}+(\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1}+f_{k,t+1}^{M,M}\right)\\
&=f_{k,t+1}^{0,0}+\varphi^\xi f_{k,t+1}^{1,0}+\sum_{n=1}^{M-1}(\varphi^\xi)^n\left(\varphi^\xi f_{k,t+1}^{n+1,n}+f_{k,t+1}^{n,n}+\varphi^{-\xi}+f_{k,t+1}^{n-1,n}\right)+(\varphi^\xi)^M\left(f_{k,t+1}^{M,M}+\varphi^{-\xi}f_{k,t+1}^{M-1,M}\right)\\
&=\begin{pmatrix}1\\ \varphi^\xi\end{pmatrix}'\begin{pmatrix}f_{k,t+1}^{0,0}\\ f_{k,t+1}^{1,0}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^\xi)^n\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^\xi\end{pmatrix}'\begin{pmatrix}f_{k,t+1}^{n-1,n}\\ f_{k,t+1}^{n,n}\\ f_{k,t+1}^{n+1,n}\end{pmatrix}+(\varphi^\xi)^M\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\begin{pmatrix}f_{k,t+1}^{M-1,M}\\ f_{k,t+1}^{M,M}\end{pmatrix}\\
&=\left(\rho_{k,0}g_{t,0}^{(k)}+\sqrt{\varrho_{k,0}g_{t,0}^{(k)}}\varepsilon_{t+1,0}\right)+\sum_{n=1}^{M-1}(\varphi^\xi)^n\left(\rho_k g_{t,n}^{(k)}+\sqrt{\varrho_k g_{t,n}^{(k)}}\varepsilon_{t+1,n}\right)\dots\\
&\quad\dots+(\varphi^\xi)^M\left(\rho_{k,M}g_{t,M}^{(k)}+\sqrt{\varrho_{k,M}g_{t,n}^{(k)}}\varepsilon_{t+1,M}\right).
\end{aligned}$$

Because $\begin{pmatrix}f_{k,t+1}^{n-1,n}\\ f_{k,t+1}^{n,n}\\ f_{k,t+1}^{n+1,n}\end{pmatrix}\approx Z\rightsquigarrow\mathcal{N}\left(g_{t,n}^{(k)}\begin{pmatrix}a_k\\ b_k\\ c_k\end{pmatrix},g_{t,n}^{(k)}\Sigma_k\right)$, it follows that $\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^\xi\end{pmatrix}'\begin{pmatrix}f_{k,t+1}^{n-1,n}\\ f_{k,t+1}^{n,n}\\ f_{k,t+1}^{n+1,n}\end{pmatrix}\approx\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^\xi\end{pmatrix}'Z\rightsquigarrow$ $\mathcal{N}\left(g_{t,n}^{(k)}\begin{pmatrix}x^{-\xi}\\ 1\\ x^\xi\end{pmatrix}'\begin{pmatrix}a_k\\ b_k\\ c_k\end{pmatrix},g_{t,n}^{(k)}\begin{pmatrix}x^{-\xi}\\ 1\\ x^\xi\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^\xi\end{pmatrix}\right)=\mathcal{N}\left(g_{t,n}^{(k)}\rho_k,g_{t,n}^{(k)}\varrho_k\right)$., where $\rho_k=a_k\varphi^{-\xi}+b_k+c_k\varphi^\xi$ and $\varrho_k=a_k\varphi^{-2\xi}+b_k+c_k\varphi^{2\xi}-\rho_k^2$. The same reasoning applies for $n=M_k$ with $\rho_{k,M}=\rho_k+c(1-\varphi^\xi):=\rho_k+\widetilde{\rho}_{k,M}$ and $\varrho_{k,M}=\varrho_k-c(1-c)(1-x^{2\xi})-2cb(1-\varphi^\xi)-2ca(1-\varphi^\xi):=\varrho_k+\widetilde{\varrho}_{k,M}$. The same reasoning applies for $n=0$ with $\rho_{k,0}=\rho_k+a(1-\varphi^{-\xi}):=\rho_k+\widetilde{\rho}_{k,0}$ and $\varrho_{k,0}=\varrho_k-a(1-a)(1-x^{-2\xi})-2ab(1-\varphi^{-\xi})-2ac(1-\varphi^{-\xi}):=\varrho_k+\widetilde{\varrho}_{k,0}$.

From this, it follows that

$$
\begin{aligned}
MZ_{t+1,k}(\xi) &= \left(\widetilde{\rho}_{k,0} g_{t,0}^{(k)}\right) + \rho_k \sum_{n=0}^{M} (\varphi^{\xi})^n g_{t,n}^{(k)} + (\varphi^{\xi})^M \left(\widetilde{\rho}_{k,M} g_{t,M}^{(k)}\right) + \sigma_{t,k}(\xi)\varepsilon_{t+1} \\
&= \rho_k(\xi) MZ_{t,k}(\xi) + O_{t,k}^{M}(\xi) + \sigma_{t,k}(\xi)\varepsilon_{t+1},
\end{aligned}
$$

where $O_{t,k}^{M}(\xi) = \widetilde{\rho}_{k,0} g_{t,0}^{(k)} + (\varphi^{\xi})^M \widetilde{\rho}_{k,M} g_{t,M}^{(k)}$. Because the $\varepsilon_{t+1,n}$ are independent across n, the variance of $\sigma_{t,k}(\xi)\varepsilon_t$ is the sum of the variances of $\sqrt{\varrho_k g_{t,n}^{(k)}}\varepsilon_{t+1,n}$; that is,

$$
\begin{aligned}
\sigma_{t,k}(\xi)^2 &= \varrho_{k,0} g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \varrho_{k,M} g_{t,n}^{(k)} \\
&= (\varrho_k + \widetilde{\varrho}_{k,0})\, g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M (\varrho_k + \widetilde{\varrho}_{k,M})\, g_{t,n}^{(k)} \\
&= \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + \sum_{n=0}^{M} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)} \\
&= \varrho_k(\xi) MZ_{t,k}(2\xi) + O_{t,k}^{\sigma}(\xi),
\end{aligned}
$$

where $O_{t,k}^{\sigma}(\xi) = \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)}$. Moreover, ε_{t+1} follows a standard normal distribution because the $\varepsilon_{t+1,n}$ are also normally distributed. □

Lemma 5 ***(Covariance of Moments of the Productivity Distribution)*** *Under Assumption 1, the covariance between the ξth moment and the ξ'th moment of the productivity distribution within sector k is given by*

$$
\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \overline{\varrho}_k(\xi,\xi') MZ_{t,k}(\xi'+\xi) + O_{t,k}^{C}(\xi,\xi'),
$$

where $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^{\xi}$ and $\overline{\varrho}_k(\xi,\xi') = a_k(1-a_k)\varphi_k^{-(\xi+\xi')} + b_k(1-b_k) + c_k(1-c_k)\varphi_k^{\xi+\xi'} - a_k b_k(\varphi_k^{-\xi} + \varphi_k^{-\xi'}) - a_k c_k(\varphi_k^{-(\xi-\xi')}\varphi_k^{\xi-\xi'}) - b_k c_k(\varphi_k^{\xi} + \varphi_k^{\xi'})$.

Proof of Lemma 5: In the proof of Lemma 4, we have

$$
MZ_{t+1,k}(\xi) = \begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}.
$$

Here, I drop the subscript k to keep the notation simpler. Let us compute the covariance between two moments of the productivity distribution in sector k:

$$
\begin{aligned}
&\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi'})^n \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi'})^M \Big(\varphi^{-} \right. \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}\right] + \sum_{n=1}^{M-1}\sum_{n'=1}^{M-1} (\varphi^{\xi})^n (\varphi^{\xi'})^{n'} \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n'-1,n'} \\ f_{k,t+1}^{n',n'} \\ f_{k,t+1}^{n'+1,n'} \end{pmatrix}\right] + \dots \\
&\dots + (\varphi^{\xi+\xi'})^M \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}\right] \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}\right] + \sum_{n=1}^{M-1} (\varphi^{\xi+\xi'})^n \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}\right] \dots \\
&\dots + (\varphi^{\xi+\xi'})^M \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}\right].
\end{aligned}
$$

where in the second line, we use the fact that $f_{k,t+1}^{\cdot,0}$ and $f_{k,t+1}^{\cdot,M}$ are independent of the $f_{k,t+1}^{\cdot,n}$ for any $0 < n < M$, and in the third line, that $f_{k,t+1}^{\cdot,n}$ are independent across n. Using the fact that $\mathbb{C}ov[A'X, B'Y] = A'\mathbb{C}ov[X,Y]B$

for vectors A and B and that random vectors X and Y are of appropriate size, we have

$$\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] =$$
$$\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \mathbb{C}ov_t\left[\begin{pmatrix} f^{0,0}_{k,t+1} \\ f^{1,0}_{k,t+1} \end{pmatrix}; \begin{pmatrix} f^{0,0}_{k,t+1} \\ f^{1,0}_{k,t+1} \end{pmatrix}\right]\begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix} + \sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \mathbb{C}ov_t\left[\begin{pmatrix} f^{n-1,n}_{k,t+1} \\ f^{n,n}_{k,t+1} \\ f^{n+1,n}_{k,t+1} \end{pmatrix}; \begin{pmatrix} f^{n-1,n}_{k,t+1} \\ f^{n,n}_{k,t+1} \\ f^{n+1,n}_{k,t+1} \end{pmatrix}\right]\begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix} \dots$$
$$\dots + (\varphi^{\xi+\xi'})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \mathbb{C}ov_t\left[\begin{pmatrix} f^{M-1,M}_{k,t+1} \\ f^{M,M}_{k,t+1} \end{pmatrix}; \begin{pmatrix} f^{M-1,M}_{k,t+1} \\ f^{M,M}_{k,t+1} \end{pmatrix}\right]\begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}.$$

Using the definition of Σ, $\Sigma^{(0)}$ and $\Sigma^{(M)}$ yields

$$\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] =$$
$$g^{(k)}_{t,0}\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma^{(0)} \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix} + \sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n g^{(k)}_{t,n}\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix} + (\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \Sigma^{(M)} \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}$$

To complete the proof, note that

$$\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix} = a(1-a)\varphi^{-(\xi+\xi')} + b(1-b) + c(1-c)\varphi^{\xi+\xi'} - ab(\varphi^{-\xi} + \varphi^{-\xi'}) - ac(\varphi^{-(\xi-\xi')}\varphi^{\xi-\xi'}) - bc(\varphi^{\xi} + \varphi^{\xi'})$$
$$\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma^{(0)} \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix} = c(1-c)(1 - \varphi^{\xi'} - \varphi^{\xi} + \varphi^{\xi+\xi'})$$
$$\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \Sigma^{(M)} \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix} = a(1-a)(1 - \varphi^{-\xi'} - \varphi^{-\xi} + \varphi^{-(\xi+\xi')}),$$

which implies

$$\begin{aligned} &O^C_{t,k}(\xi,\xi') \\ &= g^{(k)}_{t,0}\left(\left(\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix} - \begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma^{(0)} \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}\right)\right) + (\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\left(\left(\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix} - \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \Sigma^{(M)} \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}\right)\right) \\ &= g^{(k)}_{t,0}\overline{\varrho_{k,0}} + (\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\overline{\varrho_{k,M}}. \end{aligned}$$

□

Proof of Proposition 7: Using Lemma 4 and the fact that $\overline{Z_{t,k}} = MZ_{t,k}\big((\varepsilon_k - 1)\gamma_k\big)$ and that $\Delta_{t,k} = \overline{Z_{t,k}}^2 MZ_{t,k}\big(2(\varepsilon_k - 1)\gamma_k\big)$, we have

$$\overline{Z_{t+1,k}} = \rho^{(Z)}_k \overline{Z_{t,k}} + o^{(Z)}_{t,k} + \sqrt{\varrho^{(Z)}_k \Delta_{t,k} + O^{(Z)}_{t,k}} \quad \overline{Z_{t,k}} \quad \varepsilon^{(Z)}_{t+1,k}$$
$$\left(\frac{\overline{Z_{t+1,k}}}{\overline{Z_{t,k}}}\right)^2 \Delta_{t+1,k} = \rho^{(\Delta)}_k \Delta_{t,k} + o^{(\Delta)}_{t,k} + \sqrt{\varrho^{(\Delta)}_k \kappa_{t,k} + O^{(\Delta)}_{t,k}} \quad \Delta_{t,k} \quad \varepsilon^{(\Delta)}_{t+1,k}$$

Finally, Lemma 5 shows the covariance $\mathbb{C}ov_t\left[\varepsilon^{(1)}_{t+1}; \varepsilon^{(2)}_{t+1}\right] \neq 0$. □