PREFERENCE IDENTIFICATION

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Abstract. An experimenter seeks to learn a subject's preference relation. The experimenter produces pairs of alternatives. For each pair, the subject is asked to choose. We argue that, in general, large but finite data do not give close approximations of the subject's preference, even when countably infinite many data points are enough to infer the preference perfectly. We then provide sufficient conditions on the set of alternatives, preferences, and sequences of pairs so that the observation of finitely many choices allows the experimenter to learn the subject's preference with arbitrary precision. The sufficient conditions are strong, but encompass many situations of interest. And while preferences are approximated, we show that it is harder to identify utility functions. We illustrate our results with several examples, including expected utility, and preferences in the Anscombe-Aumann model.

1. Introduction

Consider a subject who forms a preference over the objects, or alternatives, of some collection X. The subject participates in an experiment, in which he is presented with a sequence of pairs of alternatives. For each pair, the subject is asked to choose one of the two alternatives offered. What can an experimenter learn about the subject's preference from observing these binary comparisons? Suppose that, after every observation, the experimenter computes an estimate of the subject's preference consistent with the data

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observed up that point: the experimenter chooses a preference rationalizing the choices made by the subject in the experiment. Is the estimate a good approximation of the subject's underlying preference, for a large but finite experiment?

In this paper, depending on the setup, we provide positive and negative answers to these questions. We investigate the asymptotic behavior of preference estimates from finite experiments. It is a question of preference identification, in the classical sense of the term. We ask if one can fully identify the preference of a subject at the limit with finite data.[1]

To illustrate the key issues, consider the following simple example. Let $X \subseteq \mathbf{R}^n$ represent a set of consumption bundles. The subject has a preference, denoted $\succeq^*$, over the elements of X. Over time, the subject is presented with a choice from a set $B_k = \{x_k, y_k\}$. Together, the sets $B_1, B_2, \ldots, B_k$ form a finite experiment. The experimenter observes the subject's choice of bundle for every pair. Assume the choice is consistent with the subject's preference, so that if x is chosen over y, then $x \succeq^* y$. Note that we can only, at best, infer the preference of the subject on the set $B \equiv \cup_{k=1}^{\infty} B_k$. Thus, if the subject's preference behaves very differently outside of the set B, there is no hope to obtain a fine approximation of the subject's preference over the entire set X. Two natural conditions emerge. First, we require that $\succeq^*$ be continuous, so one can hope to approximate the preference from finitely many samples. Second, we require that the set B is dense in X, so that the observations are sufficiently spread out. And indeed, we show that, under these conditions, if one can observe the preference of the subject over the whole set B, then one can infer precisely $\succeq^*$ on X.

The two conditions, continuity of $\succeq^*$ and denseness of B, are, however, *not* enough to provide good approximations of $\succeq^*$ from *finitely* many observations. Knowledge of the preference over the infinite set B allows the experimenter to exploit the continuity assumption on the subject's preference. With finite data,

[1]Standard decision-theoretic language reserves the term *identified* for a relation between preference and utility. In that context, a model is identified if every preference relation is represented by a unique (up to some class of transformations) set of parameters. Thus, identification in this sense requires the knowledge of an entire preference relation. In this paper, we do not assume knowledge of the entire preference relation. Instead, we ask if one can learn the entire preference relation with a possibly large, but nonetheless finite data set.

continuity does not have enough bite. For example, let $X = [0, 1]$. Suppose that the preference of the subject over X is captured by the binary relation $\geq$ (greater numbers are always chosen). Consider the countable set of objects $B = \mathbf{Q} \cap (0, 1)$, and $B_1, B_2, \ldots$ an enumeration of pairs of objects of B. Then any continuous preference that agrees with $\geq$ on $\mathbf{Q}$ has 1 weakly preferred to 0. However, for any n, one can find a preference $\succeq_n$ that rationalizes the choices of the subject over $B_1, \ldots, B_n$, and yet that ranks 0 strictly above 1.

More generally, we demonstrate in Proposition 3 that one can come up with an even more startling example: no matter the subject's preference, the experimenter may end up inferring that the subject is indifferent among all alternatives. And yet, as in the example just described, she would be able to infer the subject's preference perfectly, had she access to the subject's preference over the infinite set B all at once. The example exhibits a kind of discontinuity. With infinite data in the form of B we must conclude that $x \succeq y$, but any finite data cannot rule out that $y \succ x$.

Our examples illustrate the dangers of data-driven estimation. Non-parametric estimation with finite data can behave very differently from estimation with infinite (even countable) data. To derive meaningful estimates, one must construct a theory that disciplines the preferences, and lays down the proper conditions for convergence of preference estimates.

We include three sets of results.

Our first and foremost results concern non-parametric estimation. We offer fairly general conditions so that observing sufficiently many binary choices allows one to approximate the subject's preference arbitrarily closely with *any preference that rationalizes the finite data*.

We provide two notions of rationalization, a weak and a strong one. Under strong rationalization, a rationalizing preference must reflect choices perfectly. So if one alternative is chosen over another, the preference must rank the first strictly above the second. Under weak rationalization, the first alternative must only be ranked at least as good as the second. Weak alternatives reflect the phenomenon of *partial observability* (Chambers et al. (2014)) whereby one cannot infer anything from a choice that was not made.

Under both notions of rationalization, we must impose some structure on the environment and on the rationalizing preferences so as to avoid the negative results described above. Importantly, we need a notion of *objective* rationality expressed by the monotonicity of preferences. We postulate an exogenous partial ordering of the set of alternatives—for example, standard vector dominance when the set of alternatives represents consumption bundles, or stochastic dominance when it is the set of lotteries over monetary amounts—and we require that the subject's preference is monotonic with respect to that exogenous order.

With the added structure, finite-experiment rationalizable preferences converge to the subjects underlying preference. Somewhat stronger conditions are needed to obtain the result for weak rationalization (conditions that hold for preferences over Euclidean spaces, but rule out some common applications in decision theory), yet it is remarkable that convergence is at all attained for weak rationalization. We are after all inferring a lot less about the subject's preferences when we use weak, instead of strong, rationalization. Convergence is obtained for strong rationalizations under conditions that are consistent with most applications in decision theory, and with probably all experimental implementations of decision theoretic models.

The proposed results are general and relevant to a wide range of contexts. For concreteness, we illustrate their application to the special case of preferences over lotteries, dated rewards, consumption bundles, and Anscombe-Aumann acts (Anscombe and Aumann, 1963). In all these cases there is a natural objective partial order, and monotonicity seems to us as a very reasonable imposition. There are other environments in which one cannot reasonably impose any kind of monotonicity. For instance, in the literature on discrete allocation (**?** for example; or the recent literature on school choice) in which agents are assumed to choose among lotteries over finitely many heterogeneous objects, monotonicity would require that all agents agree on a ranking of the underlying objects. There are also some more subtle technical issues, even when the meaning of monotonicity is clear. We can deal with preferences on $\mathbf{R}^n$, and with preferences on lotteries over $\mathbf{R}$, but not with preferences over lotteries over $\mathbf{R}^n$.

Our second set of results concern the identification of utility functions. Given a utility representation for the agent's preference, we show that it is possible to carefully select finite-data utility rationalizations so as to approximate the subject's utility arbitrarily closely. This result again rests on monotonicity assumptions (but of a somewhat different nature, see Section 5). There is a clear difference between estimating preferences and utilities. Any preference estimate converges to the true underlying preference. For utilities we only know that a certain selection converges. This observation is important because one may want to estimate utilities of a certain functional form. There is no guarantee that such utility estimates have the correct asymptotic behavior: only that the preferences that they represent do.

Our third and final results concern the identification of preferences with infinite but countable data. We show that, when the experimenter has access to the preference of the subject over all alternatives of a countable set, then it is possible to recover perfectly the subject's preference over the entire set of alternatives X under much weaker conditions than above. We further demonstrate that, under such conditions, the experimenter can, in theory, obtain the subject's preference directly from the observation of a single choice of the subject when the subject is asked to select an object among a large, infinite set.

The remainder of the paper proceeds as follows. After reviewing the literature, we describe the model in Section 2. In Section 3, we discuss the special case of Anscombe-Aumann preferences. We provide our main results on non-parametric preferences with finite data in Section 4. In Section 5, we discuss the convergence of utility estimates. We deal with preference identification with infinite but countable data in Section 7. Finally, in Section 8, we discuss interpretations of preference relations. We relegate the proofs and more technical results (some of which may be of independent interest) in the Appendix.

Literature Review. Experimentalists and decision theorists have an obvious interest in preference estimation, but we are not aware of any study of the behavior of preference estimates from finite experiments. The long tradition of revealed preference theory from finite data (starting from Afriat (1967)) is focused on testing, not estimation. The closest study to ours seems to be the paper by Mas-Colell (1978), working with finite observations from a demand function over a finite number of goods. Mas-Colell assumes a rational demand function that satisfies a boundary condition and is "income Lipschitzian." He assumes a sufficiently rich sequence of observations, taken from an increasing sequence of budgets. Then he shows that the sequence of rationalizing preferences, each rationalizing a finite (but increasing) set of observations, converges to the unique preference that rationalizes the demand function.

There are many differences between Mas-Colell's exercise and ours, even if one restricts attention to choice over bundles of finitely-many, divisible, consumption goods. In particular, the difference in model primitives—demand instead of binary comparisons—is crucial. One cannot generally use choice from linear budgets to recreate any given binary comparison. Moreover, there is no property analogous to the boundary and Lipschitz continuity of demand in our framework. Indeed, as shown in Mas-Colell (1977), by means of an example due to L. Shapley, without these properties, preferences are not identified from demand.[2] In Mas-Colell's work, weak and strong rationalizability coincide, as he works with demand functions. We are particularly interested in partial observability.

Also with demand primitives, the recent papers by **?**, **?** and **?** provide results on the limiting behavior of finite-data utility rationalizations. These papers focus on the convergence of certain utility constructions that rationalize finite demand data. Our work is closer to Mas-Colell's, in that our main results are about the convergence of (any) rationalizing preferences. Of course there are also important differences in the primitives assumed in our paper and in the demand-theory papers.

The topology on preferences was introduced by Hildenbrand (1970) and Kannai (1970), building on the work of Debreu (1954). In our study of the mapping from utility to preference, we borrow ideas from Mas-Colell (1974) and Border and Segal (1994). In particular, the proof of the continuity of the

[2]Shapley's example also appears in Rader (1972). The example poses no problem for identification in our framework of binary comparisons. It generates non-identification of demand because two preferences have the same marginal rate of substitution at the sampled points. With binary comparisons, the differences between two such preferences are detected.

"certainty equivalent" representation is analogous to Mas-Colell's, and we take the notion of local strictness from Border and Segal, as well as their continuity result (see Theorem 20).

2. Model

2.1. **Notational conventions.** If $x, y \in \mathbf{R}^n$, then $x \geq y$ means that $x_i \geq y_i$ for $i = 1, \ldots, n$; and $x > y$ that $x \geq y$ and $x \neq y$. We write $x \gg y$ when $x_i > y_i$ for $i = 1, \ldots, n$. The *interval* $[a, b]$ denotes the set $\{z \in \mathbf{R}^n : b \geq z \geq a\}$. An *open interval* (a, b) denotes the set $\{z \in \mathbf{R}^n : b \gg z \gg a\}$.

If $A \subseteq \mathbf{R}$ is a Borel set, we write $\Delta(A)$ for the set of all Borel probability measures on A. For $x, y \in \Delta(A)$, we write $x \geq_{FOSD} y$ when x is larger than on y in the sense of *first order stochastic dominance* (meaning that $\int_A f dx \geq \int_A f dy$ for all monotone increasing, continuous and bounded functions f on A).

Let X be a set. Given a binary relation $B \subseteq X \times X$, we write $x \; B \; y$ when $(x, y) \in B$. And we say that a function $u : X \to \mathbf{R}$ *represents* B if $x \; B \; y$ iff $u(x) \geq u(y)$.

2.2. **The model.** There is an experimenter (a female) and a subject (a male). The subject chooses among alternatives in a set X of possible alternatives. We assume that X is a Polish and locally compact topological space. For example, the elements of X could consist of lotteries, consumption bundles, state-contingent payments, or state-contingent lotteries (so-called Anscombe-Aumann acts).

A *preference*, or *preference relation*, is any binary relation over X. A preference $\succeq$ is *continuous* if $\succeq \subseteq X \times X$ is closed. Unless otherwise specified, we assume that all preferences in this paper are complete, transitive, and continuous. In other words, they are closed weak orders, or weak orders for which the sets $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ are closed for all $x \in X$, see, *e.g.,* Bergstrom et al. (1976).

The subject in question has a preference $\succeq^*$ over X that the experimenter is trying to infer through a sequence of experiments. The experimenter designs an increasing sequence of finite experiments, whose purpose is to learn the subject's preference, $\succeq^*$, in the limit. In each experiment, the subject is presented with finitely many unordered pairs of alternatives drawn from X. For every pair $\{x, y\}$, the subject is asked to choose one of the two alternatives x or y. Continuity of the subject's preferences makes it viable to derive good approximations of the preference despite only having finite data. The problem of inferring a non-continuous preference from finite data is hopeless.

Let $\Sigma_\infty = \{B_i\}_{i\in\mathbf{N}}$ be the set of all pairs to be used across these experiments. Every $B_i = \{x_i, y_i\}$ is a subset of X of cardinality two. Enumerating the experiments $k = 1, 2, \ldots$, the set of pairs of the k-th experiment is denoted Σ_k, and it is assumed that $\Sigma_k = \{B_1, \ldots, B_k\}$; in other words, experiments are increasing in that the k-th experiment includes the $(k-1)$-th experiment. In the sequel, experiments are described in terms of their set of pairs Σ_k. Let $B = \cup_{k=1}^{\infty} B_k$ be the set of all alternatives that are used over all the experiments.

We make two assumptions on these experiments. First, we assume that the experimenter can eventually learn the subject's preference over any finite subset of B; specifically, we assume that all subsets of B of cardinality two are in Σ_∞. Second, we assume that B is dense in X. The denseness assumption is substantial, but unavoidable given our focus on nonparametric estimation. The purpose of the assumption is to provide the experimenter with a sample of alternatives spread enough over X so as to be in the capacity of inferring aspects of the preference over alternatives not used in the experiments.

The subject's behavior in the experiment is coded in a *choice function.* Formally, *a choice function of order* k is a map $c : \Sigma_k \to 2^B$ such that, for all $B_i \in \Sigma_k$, $\emptyset \neq c(B_i) \subseteq B_i$. It captures the observations of a subject who participates in finite experiment Σ_k. Let $\mathcal{C}^k$ be the set of all choice functions of order k.

The behavior of a subject across the entire sequence of experiments is captured by the *choice sequence*, defined as a function $c : \mathbf{N} \to \bigcup_k \mathcal{C}^k$ such that for all k, $c^k \in \mathcal{C}^k$, and for all $k < l$, $c^l(B_i) = c^k(B_i)$ for every $B_i \in \Sigma_k$. This last requirement conveys a consistency of behavior across experiments: a subject responds in the same way to a binary choice, no matter the experiment. We denote by $\mathcal{C}$ the set of all choice sequences. If, for two choice sequences c and c', it is the case that for all k and all $B_i \in \Sigma_k$, $c^k(B_i) \subseteq c'^k(B_i)$, then we write $c \sqsubseteq c'$. Observe that, if $c \sqsubseteq c'$, then $c^k(B_i) = c'^k(B_i)$ except possibly when

$c'^k(B_i) = B_i = \{x_i, y_i\}$, in which case we may have $c^k(B_i) = x_i$ or $c^k(B_i) = y_i$. In words, we allow for c and c' to be different and yet to be associated with observations of the same subject, the key distinction being in how indifference is treated.

Of particular interest are choices that can be *rationalized* by a preference. Given a preference $\succeq$, *the choice function of order k generated by* $\succeq$ is defined by

$$c_{\succeq}(B_i) = \arg\max_{B_i}\{x \in B_i : x \succeq y \text{ for all } y \in B_i\},$$

for $B_i \in \Sigma_k$. The choice sequence generated by $\succeq$ is defined analogously.

Thus, the choice sequence generated by the subject's preference reflects both strict comparisons as well as indifferences. In practice, however, the experimenter may not be able to properly infer the indifference of the subject regarding two alternatives. The difficulty arises, for example, when the experimenter offers the subject his preferred alternative. In this case, the experimenter would typically require that the subject selects only one of the two alternatives presented to him. Such situations, in which the experimenter cannot commit to being able to see all potentially chosen elements, are referred to *partial observability* (Chambers et al., 2014), in contrast to full observability in which the experimenter is able to elicit the subject's indifference between alternatives.

To handle situations of partial observability as well as situations of full observability, we discuss two notions of rationalization. The first notion is weak. It expresses the idea that the experimenter is not willing to commit to interpreting observed choices as the *only* potential choices made by the subject. For example, if the experimenter observes that the subject chooses x when presented the pair $\{x, y\}$, she may not be willing to infer that $x \succ^* y$, as it may be that $x \sim^* y$ but the subject simply did not choose y. This notion of weak rationalization is used, for example, by Afriat (1967) in the context of consumer theory (for more details on this notion, see, for example, Chambers and Echenique (2016)). Weak rationalization is formally defined as follows.

Definition 1. A preference $\succeq$ *weakly rationalizes* a choice function c of order k if $c(B_i) \subseteq c_{\succeq}(B_i)$ for all $B_i \in \Sigma_k$. A preference $\succeq$ *weakly rationalizes* a choice sequence c if $c \sqsubseteq c_{\succeq}$.

The second notion is a stronger one. It requires that the experimenter observes all potential choices that can be made by the subject. It is closer in spirit to the notion used in classical choice theory, and in particular, used in Richter (1966, 1971).

Definition 2. A preference $\succeq$ *strongly rationalizes* a choice function c of order k if $c(B_i) = c_{\succeq}(B_i)$ for all $B_i \in \Sigma_k$. A preference $\succeq$ *strongly rationalizes* a choice sequence c if $c = c_{\succeq}$.

A basic motivation for our analysis is the observation that, absent any discipline on the rationalizing preferences, it is impossible to achieve desirable asymptotic properties of the finite-experiment estimates. In fact, there always exists continuous and complete rationalizing preferences that converge to the *total indifference preference relation* $X \times X$, the preference relation by which every alternative is indifferent to every other alternative.

Proposition 3. *Let $X = [a, b] \subseteq \mathbf{R}^n$, where $a \ll b$, and let $\succeq^*$ be a continuous preference relation on X. There is a sequence $\{\succeq_k\}$ of continuous preference relations on X such that, for each k, $\succeq_k$strongly rationalizes the choice function of order k generated by $\succeq^*$, and such that*

$$\succeq_k \to X \times X.$$

Proof. Denote by (a', b') the open interval $\{z \in \mathbf{R}^n : a' \ll z \ll b'\}$. For each k, let $u_k : \cup_{l=1}^k B_l \to [0, 1]$ be a utility representation of $\succeq^*$ on $\cup_{l=1}^k B_l$.

For each k, let $\{[a_i, b_i]\}_{i=1}^{n_k}$ be a sequence of intervals in $\mathbf{R}^n$ with the properties that a) $[a, b] \subseteq \cup_{i=1}^{n_k}[a_i, b_i]$, b) $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ for $i \neq j$, c) each element of $\cup_{l=1}^k B_l$ is contained in a set (a_i, b_i), and no two elements of $\cup_{l=1}^k B_l$ are contained in the same, and d) $[a_i, b_i]$ is contained in some ball of radius $(2k)^{-1}$.[3]

For each interval $[a_i, b_i]$ there is a continuous function f_i such that $f(x) = 0$ for all $x \in [a_i, b_i] \setminus (a_i, b_i)$, $f(x) = u_k(x)$ if $x \in (a_i, b_i) \cap \cup_{l=1}^k B_l$, $\sup\{f(x) : x \in [a_i, b_i]\} = 2$ and $\inf\{f(x) : x \in [a_i, b_i]\} = -2$. Let $u_k^* : [a, b] \to \mathbf{R}$ be

[3]It is obvious that such a sequence exists. First, it is immediate that it exists for $n = 1$. For $n > 1$ project each B_k onto each of its coordinate and carry out the one-dimensional construction (choosing a sufficiently small radius for the balls covering each interval). Then take the cartesian product of each one-dimensional interval.

the function that coincides with f_i on each $[a_i, b_i]$. Let $\succeq_k$ be the preference relation represented by u_k^*, and note that $\succeq_k$ strongly rationalizes the choice function of order k generated by $\succeq^*$, and is continuous.

Let $x, y \in X$. For each k, suppose that $x \in [a_i, b_i]$ for the kth sequence of subintervals. Let $x_k \in [a_i, b_i]$ be such that $u_k^*(x_k) = 2$. Note that $\|x - x_k\| < 1/k$. Similarly, suppose that $y \in [a_j, b_j]$ for the kth sequence of subintervals and let $y_k \in [a_j, b_j]$ be such that $u_k^*(y_k) = -2$. Then $x_k \succ_k y_k$. Since $(x_k, y_k) \to (x, y)$ and $x, y \in X$ were arbitrary this means that $\succeq_k \to X \times X$. $\square$

Monotone Preferences. As discussed in the introduction, and exemplified by Proposition 3, the continuity assumption on the subject's preference, and the assumption that the alternatives offered are in the limit dense, do not generally ensure convergence to the subject's preference. Proposition 3 shows that the failure of convergence can be rather dramatic. We must impose structure on the subject's preference, and on the finite-experiment rationalizations. We focus on the monotonicity of preferences.

Monotonicity relates the subject's preference to an objective, common taste across all subjects being considered. Formally, we endow the space of alternatives X with a partial order $\geq$, as a well as a strict order $>$, such that $x > y$ implies $x \geq y$. For example, $>$ may be the strict part of $\geq$ (but need not be; we may want to define $>$ on $\mathbf{R}^n$ to be the relation $\gg$). Writing $x \geq y$ is interpreted as "x is objectively at least as good as y", and $x > y$ as "x is objectively strictly better than y." The partial order $\geq$ is an exogenous order. It depends on the application and the set of alternatives. For example, it may take the form of vector dominance in commodity space, or first order stochastic dominance over a set of lotteries.

The notion of monotonicity states that a preference should agree with the exogenous order. We distinguish between the weak and strict form of monotonicity.

Definition 4. A preference $\succeq$ is *weakly monotone* if $x \geq y$ implies that $x \succeq y$. A preference $\succeq$ is *strictly monotone* if $x > y$ implies that $x \succ y$.

Observe that the preferences $\succeq_k$ constructed in Proposition 3 cannot be monotone. Suppose that $\succeq^*$ is a continuous preference relation, and suppose that $x \succ^* y$. In the construction in Proposition 3 we obtain a sequence of rationalizations $\succeq_k$ such that in the limit y is at least as good as x. This cannot happen if each rationalizing preference is weakly monotone: $x \succ^* y$ implies that $x' \succ^* y'$ for (x', y') close enough to (x, y). Thanks to the interaction of the order and the topology on $\mathbf{R}^n$ we can find a k large enough such that there are $\{x'', y''\} \in \Sigma_k$ (meaning alternatives offered in the kth finite experiment) with $x' \geq x''$ and $y'' \geq y'$, and where (x'', y'') is also close to (x, y). If $\succeq_k$ is monotone then we have $x' \succeq_k x''$ and $y'' \succeq y'$. But if $\succeq_k$ strongly rationalizes the choices made at the kth experiment, then $x'' \succ_k y''$. So we have to have $x' \succ_k y'$ for any (x', y') close enough to (x, y).

Convergence of Preferences. To speak about the approximation of the subject's preference, one must introduce a notion of convergence on the space of preferences. We use *closed convergence*, and endow the space of preference relations with the associated topology. The use of closed convergence for preference relations was initiated by the work of Kannai (1970) and Hildenbrand (1970), and has become standard since then.

One primary reason to adopt closed convergence is to capture the property that agents with similar preferences should have similar choice behavior—a property that is necessary to be able to learn the preference from finite data. Specifically, under the assumptions we use for most of our results, the topology of closed convergence is the smallest topology for which the sets

$$\{(x, y, \succeq) : x \succ y\}$$

are open (see Kannai (1970) Theorem 3.1). The desired continuity of choice behavior is expressed by the fact that sets of the form $\{(x, y, \succeq) : x \succ y\}$ are open. The topology of closed convergence being the smallest topology with this property is a natural reason for adopting it.

The following characterization of closed convergence for the context of preference relations is useful:

Lemma 5. *Let $\succeq_n$ be a sequence of preference relation, and let $\succeq$ be a preference relation. Then $\succeq_n \to \succeq$ in the topology of closed convergence if and only if, for all $x, y \in X$,*

(1) *$x \succeq y$ implies that for any neighborhood V of (x, y) in $X \times X$ there is N such that for all $n \geq N$, $\succeq_n \cap V \neq \varnothing$;*
(2) *if, for any neighborhood V of (x, y) in $X \times X$, and any N there is $n \geq N$ with $\succeq_n \cap V \neq \varnothing$, then $x \succeq y$.*

The following lemma plays an important role in the approximation results.

Lemma 6. *The set of all continuous binary relations on X, endowed with the topology of closed convergence, is a compact metrizable space.*

Proof. See Theorem 2 (Chapter B) of Hildenbrand (2015), or Corollary 3.95 of Aliprantis and Border (2006). □

In particular, we shall denote the metric which generates the closed convergence topology by δ_C. Recall that X is metrizable, say with metric d. When X is compact, one can choose δ_C to be the Hausdorff metric on subsets of $X \times X$ induced by d. On the other hand, if X is only locally compact, then δ_C may be chosen to coincide with the Hausdorff metric on subsets of $X_\infty \times X_\infty$, where X_∞ is the one-point compactification of X together with some metric generating X_∞. See Aliprantis and Border (2006) for details.

3. Application: Anscombe-Aumann Preferences over monetary lotteries.

Let Ω be a finite nonempty set of *states of the world*. Let $\Delta([a, b])$ be the set of all Borel probability measures over the closed interval $[a, b] \subseteq \mathbf{R}$. We interpret $[a, b]$ as a set of monetary payoffs, and the elements of $\Delta([a, b])$ as lotteries of monetary payoffs. An *Anscombe-Aumann act* is a state-contingent monetary lottery, it maps elements from Ω to $\Delta([a, b])$. Let the set of alternatives X be the set $\Delta([a, b])^\Omega$ of all Anscombe-Aumann acts.

Endow X with the product weak* topology, and consider the partial order on X obtained as the product of the first-order stochastic dominance order on $\Delta([a, b])$. Formally, we write $x \geq y$ if, for any $\omega \in \Omega$, $x(\omega) \geq_{FOSD} y(\omega)$.

Then, the following result obtains.

Theorem 7. *Suppose $\succeq^*$ is a strictly monotone preference relation. For every $k = 1, 2, \ldots$, let $\succeq_k$ be a strictly monotone preference relation which strongly rationalizes the choice function of order k generated by $\succeq^*$. Then*

- $\succeq_k \to \succeq^*$ *(in the topology of closed convergence);*
- *for any utility representation* u^* *of* $\succeq^*$*, there exist utility representations* u_k *of* $\succeq_k$ *such that* $u_k \to u^*$ *(in the topology of compact convergence).*

Theorem 7 is interesting for what it says, but also for what it does not say. The theorem says that, if we assume that the data are generated by a (well behaved) preference $\succeq^*$, then *any* "finite sample rationalization" $\succeq^k$ is guaranteed to converge to the generating preference. So estimates have the correct "large sample" properties. In particular, one may be interested in a specific theory of choice, such as max-min or Choquet expected utility. So if the subject's $\succeq^*$ is max-min, or Choquet, one can choose rationalizing preferences to conform to the theory, and the limit will uniquely identify the subject's max-min, or Choquet, preference. But if one incorrectly uses rationalizing preferences outside of the theory, the asymptotic behavior will still correct the problem and uniquely identify $\succeq^*$ in the limit. The theorem also says that there are certain utility representations u^k that will be correct asymptotically.

Note, however, what the theorem does not say. First, the estimates $\succeq^k$ are guaranteed to converge to the generating preferences $\succeq^*$, *when the generating preference is known to exist.* If one simply estimates the preferences $\succeq^k$, these may fail to converge to a well-behaved preference. We present two examples to this effect in Section 8. That said, under certain conditions (that unfortunately are not satisfied in the Anscombe-Aumann setting), the "size" of the set of rationalizing preferences shrinks as k growth; see Theorem 10.

Second, Theorem 7 does not say that one can choose u^k arbitrarily. Any estimated rationalizing preference will converge to the preferences rationalizing the utility, but basing the estimation on utilities is more complicated because it is not clear that any utility representation of $\succeq^*$ will have the right limit, or even converge at all.

This brings us to the role of identification theorems in decision theory. It is common to show that a model is identified, and argue that this enables the empirical recovery of utility parameters from observed behavior. The ideas behind Theorem 7 imply that more is required. Specifically, suppose that $\mathcal{U}$ is the set of all continuous and strictly monotone functions $u : X \to \mathbf{R}$. Denoting by $\mathcal{R}^{\mathrm{mon}}$ the set of all continuous and strictly monotone preference relations, let $\Phi : \mathcal{U} \to \mathcal{R}^{\text{mon}}$ be the function that sends each $u \in \mathcal{U}$ into the preference it represents. When we show that Φ is an open map (Theorem 19), we show that $\succeq_k \to \succeq^*$ and $u^* \in \Phi^{-1}(\succeq^*)$ imply that one can choose $u_k \in \Phi^{-1}(\succeq_k)$ with $u_k \to u^*$.[4] In fact, we show that if $\simeq$ is the equivalence relation on $\mathcal{U}$ whereby two utility functions are equivalent if they are ordinally equivalent (*i.e.*, $u \simeq u'$ iff $\Phi(u) = \Phi(u')$), then $\Phi : \mathcal{U}/\simeq \to \mathcal{R}^{\text{mon}}$ is a homeomorphism (Theorem 21).

Many results on identification in decision theory can be phrased in the following terms. There are subsets $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{R}' \subseteq \mathcal{R}^{\text{mon}}$, and an equivalence relation $\simeq'$ on $\mathcal{U}'$ such that Φ is a bijection from $\mathcal{U}'/\simeq'$ onto $\mathcal{R}'$. Our results suggest that this is not enough to conclude that empirical estimates will have the correct "large-sample behavior."

4. Main Results

In this section, we present our results on the asymptotic behavior of preference estimates based on finite data.

For our first result, we must define two notions. We say that a preference relation $\succeq$ is *locally strict* if for every $x, y \in X$ with $x \succeq y$, and every neighborhood V of (x, y) in $X \times X$ there is $(x', y') \in V$ with $x' \succ y'$. We say that the order $>$ on X has *open intervals* if $\{(x, y) : x > y\}$ is an open subset of $X \times X$. Our first main result gives conditions of convergence of preferences that weakly rationalize the experimental observations.

Theorem 8. *Suppose that*

(1) *the subject's preference $\succeq^*$ is continuous and strictly monotone,*
(2) *the strict order $<$ has open intervals,*
(3) *every continuous and strictly monotone preference relation is locally strict.*

Let $c \sqsubseteq c_{\succeq^*}$ be a choice sequence, and let $\succeq_k$ be a continuous and strictly monotone preference that weakly rationalizes c^k. Then, $\succeq_k \to \succeq^*$ in the closed convergence topology.

[4] The property of being an open map is close to being necessary for the result: see Theorem 4.2 of Siwiec (1971).

Remark 9. The assumption that $\succeq^*$ and $\succeq_k$ are transitive is not needed. Instead, each of these only needs to be continuous, strictly monotone, and complete.

Theorem 8 requires the existence of $\succeq^*$. However, even if existence of this object is not supposed, we can still "bound" the set of rationalizations to an arbitrary degree of precision. This is the content of the next result.

For a choice sequence c, let $\mathcal{P}^k(c)$ be the set of continuous and strictly monotone preferences that weakly rationalize c^k. For a set of binary relations S, define $\mathrm{diam}(S) = \sup_{(\succeq,\succeq')\in S^2} \delta_C(\succeq,\succeq')$ to be the diameter of S according to the metric δ_C which generates the topology on preferences.

Theorem 10. *Suppose that $<$ has open intervals. Let c be a choice sequence, and suppose that each strictly monotone continuous preference is also locally strict. Then one of the following holds:*

(1) *There is k such that $\mathcal{P}^k(c) = \varnothing$.*
(2) $\lim_{k\to\infty}$ *diam*$(\mathcal{P}^k(c)) \to 0$.

That is, either a choice sequence is eventually not weakly rationalizable by a strictly monotone preference, *or*, the set of rationalizations becomes arbitrarily small.

Remark 11. As for Theorem 8, Theorem 10 can dispense with the notion of transitivity. In this case, we would define $\mathcal{P}^k(c)$ to be the set of (potentially nontransitive) complete, continuous, and strongly monotone relations weakly rationalizing c^k.

Our second result applies to preferences that strongly rationalize the experimental observations. To state the result, we define two other notions. We say that the set X, together with the collection of finite experiments Σ_∞, has the *countable order property* if for each $x \in X$ and each neighborhood V of x in X there is $x', x'' \in B \cap V$ with $x' \leq x \leq x''$. We say that X has the *squeezing property* if for any convergent sequence $\{x_n\}_n$ in X, if $x_n \to x^*$ then there is an increasing sequence $\{x'_n\}_n$, and an a decreasing sequence $\{x''_n\}_n$, such that $x'_n \leq x_n \leq x''_n$, and $\lim_{n\to\infty} x'_n = x^* = \lim_{n\to\infty} x''_n$.

Theorem 12. *Suppose that*

(1) *the subject's preference* $\succeq^*$ *is weakly monotone,*
(2) (X, Σ_∞) *has the countable order property, and* X *the squeezing property.*

Let $\succeq_k$ *be a continuous and weakly monotone preference that strongly rationalizes the choice function of order* k *generated by* $\succeq^*$. *Then,* $\succeq_k \to \succeq^*$ *in the closed convergence topology.*

The countable order and squeezing properties are technical but not vacuous. Importantly, as stated below in Proposition 13, they are satisfied for two common cases of interest, which allows us to obtain the first part of Theorem 7 as a direct consequence of Theorem 12.

Proposition 13. *If either*
(1) *the set of alternatives* X *is* $\mathbf{R}^n$ *endowed with the order of weak vector dominance, or*
(2) *the set of alternatives* X *is* $\Delta([a, b])$ *endowed with the order of weak first-order stochastic dominance,*

then X *has the squeezing property, and there is* Σ_∞ *such that* (X, Σ_∞) *has the countable order property.*

One key element behind the above two results is a natural order on the sets of possible alternatives. Via monotonicity, the order adds structure to the families of preferences under consideration. Crucially, the order also relates to the topology on the set X.

Section 3 applies Theorem 12 to preferences over Anscombe-Aumann acts. We can obtain a similar result for other environments, in particular, for environments in which the alternatives can be represented as vectors in some Euclidean space. We emphasize three domains of applications: lotteries over a finite prize space, dated rewards, and consumption bundles.

Example 14. First consider lotteries over a finite set of prizes. Let Π be a finite *prize space*. The objects of choice are the elements of $X = \Delta(\Pi)$. Fix a strict ranking of the elements of Π, and enumerate the elements of Π so that π_1 is ranked above π_2, which is ranked above π_3, and so on. Then the elements of X can be ordered with respect to first-order stochastic dominance:

x is larger than y in this order if the probability of each set $\{\pi_1, \ldots, \pi_k\}$ is at least as large under x than under y, for all $k = 1, \ldots, |\Pi|$. A preference over X is monotone if it always prefer larger lotteries over smaller ones.[5]

Imagine choices generated by an expected utility preference $\succeq^*$. The fact that $\succeq^*$ is of the expected utility family implies that there are rationalizing expected utility preference $\succeq_k$, for each finite experiment k. Then our theorems ensure that these converge to $\succeq^*$. Of course the same would be true of any (monotone and continuous) rationalizing preference: any mode mis-specification would be corrected in the limit. Any arbitrary sequence of rationalization has $\succeq^*$ as its limit.

Example 15. In second place, we can apply our theory to intertemporal choice. Specifically to the choice of *dated rewards* (Fishburn and Rubinstein (1982)). The set of elements of choice is $\mathbf{R}^2_+$. A point $(x, t) \in \mathbf{R}^2_+$ is interpreted as a monetary payment of x delivered on date t. Endow $\mathbf{R}^2_+$ with the order $\leq^i$ whereby $(x, t) \leq^i (x', t')$ if $x \leq x'$ and $t' \leq t$. Monotonicity of preferences means that more money earlier is preferred to less money later.

Now one can postulate a preference $\succeq^*$ such that $(x', t') \succeq^* (x, t)$ iff $\delta^t v(x) \leq \delta^{t'} v(x')$, for some $\delta \in (0, 1)$ and a strictly increasing function $v : \mathbf{R}_+ \to \mathbf{R}$. This means that $\succeq^*$ follows the exponential discounting model. Again, any finite experiment would be rationalizable by exponential preference, and these would converge to the limiting $\succeq^*$.

Example 16. Finally, the elements of choice can be consumption bundles in $\mathbf{R}^n$. A preference over such consumption bundles is monotone if it prefers larger bundles over smaller ones. Imagine a preference $\succeq^*$ represented by, say, a Cobb-Douglas utility. Finite sample estimates would then converge to the preference $\succeq^*$.

5. Identification of Utility Functions

In this section, we investigate the relation between preferences and utility. Preferences remain topologized with the closed convergence topology. We study continuous utility representations, and ask when the identification of a

[5]The objective order on Π is not really needed in this case; see Example 23. The point of the example is to illustrate Theorem 8.

preference allows the identification of a utility (or conversely). We show that if we endow the set of continuous utility functions with the topology of uniform convergence on compacta, then convergence in one sense is equivalent to convergence in the other. Formally, we establish that there is a homeomorphism between the two spaces (when we identify two utility functions representing the same preference relation).

Throughout this section, the space of possible alternatives X is connected (and remains a locally compact Polish space, as described in our model). Connectedness is imposed so that every continuous preference admits a continuous representation, as in Debreu (1954).

We denote by $\mathcal{U}$ the set of strictly increasing and continuous utility functions on X. Similarly, $\mathcal{R}^{\text{mon}}$ denotes the set of preferences which are strictly monotone and continuous.

Suppose the existence of a set $M \subseteq X$, satisfying the following conditions:

- M has at least two distinct elements; M is connected and totally ordered by $<$. In other words $x, y \in M$ and $x \neq y$ implies $x < y$ or $y < x$.
- For any $m \in M$ and any neighborhood U of m in X there is $\underline{m}, \overline{m} \in M$, with
$$m \in [\underline{m}, \overline{m}] \subseteq U.$$
Moreover if m is not the largest element of M we can choose $\overline{m}$ such that $m < \overline{m}$, and if m is not the smallest element we can choose $\underline{m}$ such that $\underline{m} < m$.
- Any bounded sequence in X is bounded by elements of M. That is, for any bounded sequence $\{x_n\}$ there are $\underline{m}$ and $\overline{m}$ and k large so that $\underline{m} \leq x_n \leq \overline{m}$.

Let $\Phi : \mathcal{U} \to \mathcal{R}^{\text{mon}}$ such that $\Phi(u)$ is the preference represented by $u \in \mathcal{U}$.[6]

We provide two examples below, that demonstrate the property just mentioned for the case of alternatives of the form $X = \Delta([a, b])$ and $X = \Delta([a, b])^n$.

Example 17. Let $X = \Delta([a, b])$ be the set of Borel probability distributions on a real compact interval $S = [a, b] \subseteq \mathbf{R}$. Endow X with the weak* topology

[6]That is, $x\ \Phi(u)\ y$ if and only if $u(x) \geq u(y)$.

and let $\leq$ be first-order stochastic dominance. Observe that X is compact, metrizable, and separable (Theorems 15.11 and 15.12 of Aliprantis and Border (2006)). Observe also that X has the countable order property (see Lemma 32 in Appendix B).

Let $<$ be the strict part of $\leq$. Identify S with degenerate probability distributions, so that $s \in S$ denotes the element of X that assigns probability 1 to $\{s\}$, say δ_s. Let $M = S$. The relative topology on S coincides with the usual topology, so S is connected. Note that $a \leq x \leq b$ for any $x \in X$.

Let $m \in M$ and U be a neighborhood of m in X. For each $x \in X$, let F^x be the cdf associated to x. Choose ε such that the ball $B_\varepsilon(m)$ (in the Levy metric) with center m and radius ε is contained in U. Let $\varepsilon' < \varepsilon$. Then if $y \in [m - \varepsilon', m + \varepsilon']$ we have that

$$F^y(s-\varepsilon) - \varepsilon \leq F^{m-\varepsilon'}(s-\varepsilon) - \varepsilon < 1 = F^m(s) \text{ if } s - \varepsilon \geq m - \varepsilon'$$
$$F^y(s-\varepsilon) - \varepsilon \leq F^{m-\varepsilon'}(s-\varepsilon) - \varepsilon = -\varepsilon < F^m(s) \text{ if } s - \varepsilon < m - \varepsilon'$$

Similarly,

$$F^m(s) = 0 < F^{m+\varepsilon}(s+\varepsilon) + \varepsilon \leq F^y(s+\varepsilon) + \varepsilon \text{ if } s + \varepsilon \leq m + \varepsilon'$$
$$F^m(s) < 1 + \varepsilon = F^{m+\varepsilon'}(s+\varepsilon) + \varepsilon \leq F^y(s+\varepsilon) + \varepsilon \text{ if } s + \varepsilon > m + \varepsilon'.$$

These inequalities mean that $y \in B_\varepsilon(m)$. Thus $[m - \varepsilon', m + \varepsilon'] \subseteq U$, as y was arbitrary.

Example 18. Let Ω be a nonempty set such that $|\Omega| < +\infty$. Suppose Ω represents a set of *states of the world*. Then $\Delta([a,b])^\Omega$, endowed with the product weak* topology, and ordered by the product order, of Ω copies of first order stochastic dominance, represents the set of Anscombe-Aumann acts, Anscombe and Aumann (1963). Let $S = \{(\delta_s, \ldots, \delta_s) : s \in [a,b]\}$; the constant acts whose outcomes are degenerate lotteries. Let $M = S$, as in the previous example; and all topological properties satisfied there are also satisfied here.

The following generalizes results derived originally by Mas-Colell (1974), who worked with $\mathbf{R}^n_+$.

Theorem 19. *Φ is an open map.*

Theorem 20. *(Border and Segal (1994) Thm 8) Let (X, d) be a locally compact and separable metric space and $\mathcal{R}$ be the space of continuous preference relations on X, endowed with the topology of closed convergence. If $\succeq_u = \Phi(u)$ is locally strict, then Φ is continuous at u. If M has no isolated points, and Φ is continuous at u, then $\succeq_u$ is locally strict.*

Define an equivalence relation $\simeq$ on $\mathcal{U}$ by $u \simeq v$ if there exists $\varphi : \mathbf{R} \to \mathbf{R}$ strictly increasing for which $u = \varphi \circ v$. Then let $\mathcal{U}/\simeq$ denote the set of equivalence classes of $\mathcal{U}$ under $\simeq$ endowed with the quotient topology; the equivalence class of $u \in \mathcal{U}$ is written $[u]$. The map $\hat{\Phi} : \mathcal{U}/\simeq \to \mathcal{R}^{\text{mon}}$ is defined in the natural way, via $\hat{\Phi}([u]) = \Phi(u)$.[7]

Theorem 21. *$\hat{\Phi}$ is a homeomorphism.*

6. Non-monotone preferences and local strictness

In response to Section 8.1, we show how we can leverage compactness results from the theory of functions to establish the existence of a rationalizing preference in the limit. Let $\mathcal{V}$ be a compact set of continuous functions in the topology of compact convergence, and let $\Phi(\mathcal{V})$ denote the image of $\mathcal{V}$ under Φ, so that $\Phi(u)$ is the preference represented by u.

Theorem 22. *Suppose $\mathcal{V}$ is compact, and that all $\succeq \in \Phi(\mathcal{V})$ are locally strict. Let c be a choice sequence, and let $\succeq_k \in \mathcal{V}$ weakly rationalize c^k. Then, there exists $\succeq^* \in \mathcal{V}$ such that $\succeq_k \to \succeq^*$ in the closed convergence topology. Furthermore, if $\succeq'_k$ also weakly rationalizes c^k, then $\succeq'_k \to \succeq^*$.*

Theorem 22 implies that one can some times obtain asymptotically obtain utility rationalizations drawn from $\mathcal{V}$. In particular, when $\mathcal{V}$ is compact, $\Phi(\mathcal{V})$ consists of locally strict preferences, and Φ is a homeomorphism then $\Phi^{-1}(\succeq_k) \in \mathcal{V}$ converges to a utility for $\succeq^*$ in $\mathcal{V}$. One application of this kind is in Example 23.

Example 23. Let X be a finite set, and let $\Delta(X)$ be the lotteries on X (topologized as elements of Euclidean space). Consider the set of nonconstant

[7] Observe that this function is well-defined. If $v \in [u]$, then there is strictly increasing φ for which $v = \varphi \circ u$, hence v and u represent the same preference.

expected utility preferences. Then the hypotheses of Theorem 22 hold here. To see this, observe that the set of nonconstant von Neumann-Morgenstern utility indices is homeomorphic to the set

$$S = \{u \in \mathbf{R}^X : \sum_x u_x = 0, \|u\| = 1\}.$$

It is straightforward to see that the map $\phi : S \to C(\Delta(X))$ given by $\phi(u)(p) = \sum_x u_x p(x)$ is continuous. So, let $\mathcal{V} = \phi(S)$ which is compact; then the set $\Phi(\mathcal{V})$ is the set of nontrivial expected utility preferences. Finally, observe that each nonconstant expected utility preference is locally strict. For, if $\succeq$ is nonconstant, then there are $p, q \in \Delta(X)$ for which $p \succ q$. Then for any $r \succeq s$, for any $\alpha > 0$, $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)s$. Choose α small to be within any neighborhood of (r, s).

Next, Example 24 allows for an infinite set of prices, but restricts von Neumann-Morgenstern utilities to have lower and upper Lipschitz bounds.

Example 24. We can consider $\mathbf{R}^n_+$, and a class of utility functions $\mathcal{U}^b_a$, where $a, b \in \mathbf{R}$ with $0 < a < b$.

$$\mathcal{U}^b_a = \{u \in C(\mathbf{R}^n_+) : \forall i \wedge \forall (x_i < y_i), a(y_i - x_i) \leq u(y_i, x_{-i}) - u(x_i, x_{-i}) \leq b(y_i - x_i)\}.$$

Observe that $\mathcal{U}^b_a \subseteq \mathcal{U}$, and consists of those members satisfying a certain Lipschitz property (namely, Lipschitz boundedness above and below). By the Arzela-Ascoli Theorem (see Dugundji (1966), Theorem 6.4), $\mathcal{U}^b_a$ is compact. Furthermore, each $\succeq \in \Phi(\mathcal{U}^b_a)$ is locally strict, as it is strictly monotonic.

7. Infinite and Countable Data

In this section, we propose two sufficient conditions that enable the recovery of the subject's preference from its restriction to a countable set of data points.

In the first result, we show that, if we can observe a subrelation of a locally strict and continuous binary relation on a dense set, then we can infer the entire binary relation.

Theorem 25. *Suppose that $\succeq$ and $\succeq'$ are two complete and continuous binary relations. Suppose that $\succeq'$ is locally strict, and let $B \subseteq X$ be dense. If $\succeq|_{B\times B} \subseteq \succeq'|_{B\times B}$, then $\succeq = \succeq'$.*

The second result makes no restriction on the preferences other than continuity, but requires the underlying space of alternatives to be connected.

Theorem 26. *Suppose that $\succeq$ and $\succeq'$ are two continuous preference relations. Suppose X is connected, and let $B \subseteq X$ be dense. If $\succeq|_{B\times B} = \succeq'|_{B\times B}$, then $\succeq = \succeq'$.*

A classical tool, attributed to Allais (see Allais (1953)) allows one to elicit multiple choices with one suitably randomized choice. Roughly, one uses a randomization device whose outcome is a choice set, and asks a subject to announce what she would choose *ex-ante* from each of the sets in the support in the distribution. A decision maker who respects basic monotonicity postulates (see Azrieli et al. (2014)) correctly announces each of their choices.

If we can uncover an entire preference from each of these choices, then we are able to elicit an entire preference using one suitably chosen random device. Here, we do not investigate this theory in its full generality. But if there is a countable dense subset of alternatives, and a continuous preference can be inferred from its behavior on a countable dense subset, then we can utilize the Allais mechanism to uncover an entire preference with a single randomized choice. For example, we would enumerate the pairs of elements from the countable dense subset, say $B_1, B_2, \ldots$, and randomize so that each one realizes with probability 2^{-k}.

8. On the meaning of $\succeq^*$

Some economists are comfortable saying that an agent "has" a preference $\succeq^*$, and some are not. The first assume that the preference is something intrinsic to the agent, and that when presented with a choice situation the agent can access his preference and choose accordingly. The exercise in our paper gives conditions under which a finite experiment can approximate, to an arbitrary degree of precision, the underlying preference that the agent uses to make choices.

Other economists think that preferences are just choices. For people in this position, it is meaningless to speak of a preference over pairs of alternatives from which the agent never chooses. We are highly sympathetic to this view, and our paper also contributes to this interpretation. Under the right conditions (conditions that we provide in our paper) continuity "defines" preferences over X given choices over a countable subset. This is important because estimated preference provide a guide for making normative recommendations and out of sample predictions. An economist may want to estimate $\succeq^*$ so as to make policy recommendations that are in the agent's interest (in fact this is a very common use of estimated preferences in applied work). Similarly, the economist may want to use $\succeq^*$ as an input in a structural economic model, and thereby make predictions for different configurations of the model. The existence and meaning of $\succeq^*$ is then provided for by the continuity assumption.

Moreover, viewed from this angle, Theorem 10 allows us to say that the set of rationalizations can be made arbitrarily small as more and more data are observed.[8] In this manner, one can bound errors in welfare statements or out of sample predictions to an arbitrary degree of precision.

We conclude this section with two examples that illustrate the importance of postulating existence of an agent's preference: without the postulate, the inferred preference may otherwise fail to converge.

8.1. **The set of weakly monotone preference relations is not closed.** Suppose we are interested in rationality in the form of a strictly monotonic continuous preference relation. Observe that Theorems 8 and 12 hypothesize the existence of $\succeq^*$. If $\succeq^* \in \mathcal{R}^{\text{mon}}$, for example, then we know that, in the limit, rationalizing relations will be transitive if every $\succeq_k$ is. Unfortunately, we show in this section, if we do not know that $\succeq^*$ is transitive, we cannot ensure that it is, even if each $\succeq_k$ is. That is, we demonstrate a sequence $\succeq_k$ of strictly monotone preferences, where $\succeq_k \to \succeq^*$ in the closed convergence topology, but $\succeq^*$ is not transitive.

The data are rationalizable, but the rationalization requires intransitive indifference. So the properties of the rationalizations of order k cannot be preserved.

[8]This is true in spite of Section 8.1. It is true that the set of rationalizations may "shrink" to something which is not transitive, but this set is shrinking nonetheless and always contains preference relations (except in the limit).

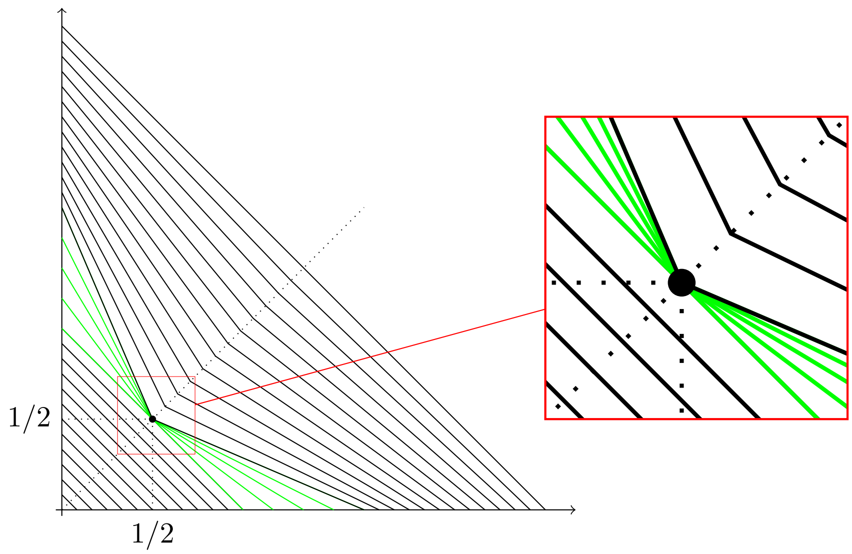

FIGURE 1. A non-transitive preference

Figure 1 exhibits a non-transitive relation. The example is taken from Grodal (1974). The lines depict indifference curves, but all the green indifference curves intersect at one point: $(1/2, 1/2)$. This makes the preference non-transitive; specifically the indifference part of the preference would be intransitive here.

Now imagine a collection of binary comparisons that do not include $(1/2, 1/2)$. Suppose that this collection is the limit of a finite number of binary comparisons, making it at most countable. There must exist a ball around $(1/2, 1/2)$ that does not include any of the comparisons. Consider the diagram in Figure 2. The preferences have been modified close to $(1/2, 1/2)$ so that transitivity holds.

This example is not particularly troubling, however. First, with finite experimentation, the violation of transitivity will never be "reached." Second, the violation here is not particularly egregious. Only transitivity of indifference is violated. This holds quite generally. It can be shown that any limit point of a sequence of preference relations must be quasitransitive, so that whenever $x \succ y$ and $y \succ z$, it follows that $x \succ z$.[9] Quasitransitive relations enjoy many

[9]The argument is in Grodal (1974), but to see this suppose that $\succeq^n \to \succeq$, where each $\succeq^n$ is a preference relation. It can be shown that $\succeq$ is complete, so suppose by means of

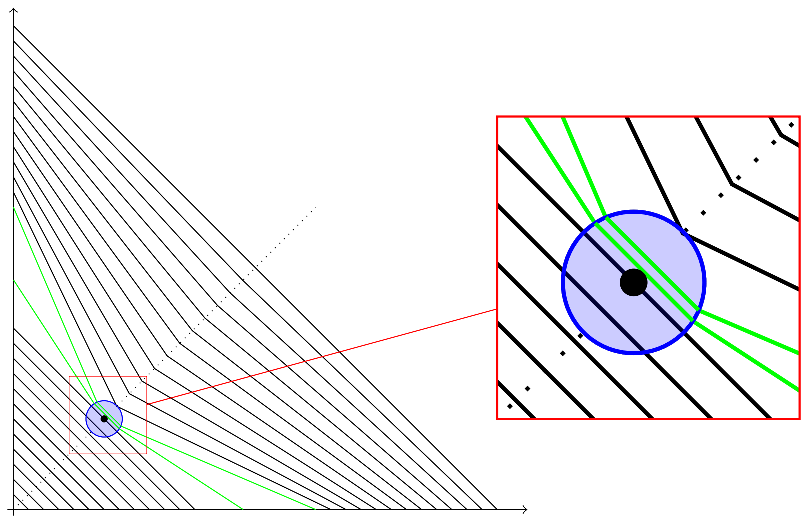

FIGURE 2. A transitive preference

of the useful properties of preferences. For example, continuous quasitransitive relations possess maxima on compact sets, see *e.g.* Bergstrom (1975).

8.2. **The set of locally strict relations is not closed.** Finally we present an example to show that the set of locally strict preference relations is not closed. Let $X = [-3,-1] \cup [1,3]$. For each n, let $u_n(x) = -(x+2)^2 + \frac{1}{n}$ on $[-3,-1]$ and $u_n(x) = (x-2)^2 - \frac{1}{n}$ on $[1,3]$. See Figure 3. The function u_n represents a locally strict relation $\succeq_n$.

Let $u^*(x)$ be the pointwise limit of u_n; i.e. $u^*(x) = -(x+2)^2$ on $[-3,-1]$ and $u^*(x) = (x-2)^2$ on $[1,3]$. The function u^* represents $\succeq^*$ which is *not* locally strict. Observe that $-2 \succeq^* 2$, but for small neighborhoods there is no strict preference.

However, it is also straightforward by checking cases to show that $\succeq_n \to \succeq^*$.

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contradiction that there are $x, y, z \in X$ for which $x \succ y$, $y \succ z$, and $z \succeq x$. So, there are x^n, z^n for which $z^n \succeq^n x^n$, $x^n \to x$, and $z^n \to z$. For each n, either $z^n \succeq^n y$ or $y \succeq^n x$, so that without loss, there is a sequence for which $z^n \succeq^n y$, *i.e.* $z \succeq y$, a contradiction.

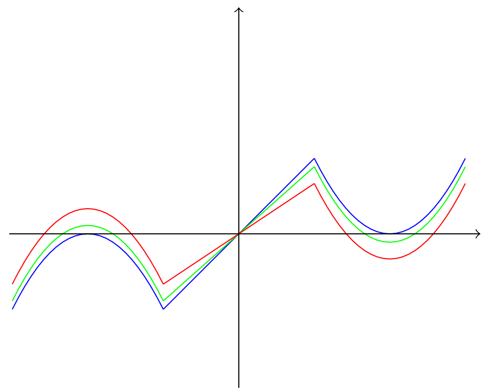

FIGURE 3. The set of locally strict preferences is not closed.

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Appendix A. About Closed Convergence

We recall below the formal definition of closed convergence, used throughout the results of this paper. Let $\mathcal{F} = \{F^n\}_n$ be a sequence of closed sets in $X \times X$. We define $\mathrm{Li}(\mathcal{F})$ and $\mathrm{Ls}(\mathcal{F})$ to be closed subsets of $X \times X$ as follows:

- $(x, y) \in \mathrm{Li}(\mathcal{F})$ if and only if, for all neighborhood V of (x, y), there exists $N \in \mathbf{N}$ such that $F^n \cap V \neq \varnothing$ for all $n \geq N$.
- $(x, y) \in \mathrm{Ls}(\mathcal{F})$ if and only if, for all neighborhood V of (x, y), and all $N \in \mathbf{N}$, there is $n \geq N$ such that $F^n \cap V \neq \varnothing$.

Observe that $\mathrm{Li}(\mathcal{F}) \subseteq \mathrm{Ls}(\mathcal{F})$. The definition of closed convergence is as follows.

Definition 27. F^n converges to F in the *topology of closed convergence* if $\mathrm{Li}(\mathcal{F}) = F = \mathrm{Ls}(\mathcal{F})$.

Appendix B. Proof of Proposition 13

The proof is implied by the following lemmas.

Lemma 28. *Let $X \subseteq \mathbf{R}^n$. If $\{x'_n\}$ is an increasing sequence in X, and $\{x''_n\}$ is a decreasing sequence, such that* $\sup\{x'_n : n \geq 1\} = x^* = \inf\{x''_n : n \geq 1\}$. *Then*

$$\lim_{n\to\infty} x'_n = x^* = \lim_{n\to\infty} x''_n.$$

Proof. This is obvously true for $n = 1$. For $n > 1$, convergence and sups and infs are obtained component-by-component, so the result follows. $\square$

Lemma 29. *Let $X \subseteq \mathbf{R}^n$. Let $\{x_n\}$ be a convergent sequence in X, with $x_n \to x^*$. Then there is an increasing sequence $\{x'_n\}$ and an a decreasing sequence $\{x''_n\}$ such that $x'_n \leq x_n \leq x''_n$, and* $\lim_{n\to\infty} x'_n = x^* = \lim_{n\to\infty} x''_n$.

Proof. Suppose that $x_n \to x^*$. Define x'_n and x''_n by

$$x'_n = \inf\{x_m : n \leq m\} \text{ and } x''_n = \sup\{x_m : n \leq m\}$$

Then it is clear that $x_n' \leq x_n \leq x_n''$, that x_n' is increasing, and that x_n'' is decreasing. Moreover,

$$\begin{aligned}\lim_{n\to\infty} x_n' &= \sup\{\inf\{x_m : n \leq m\} : n \geq 1\} \\ &= x^* \\ &= \inf\{\sup\{x_m : n \leq m\} : n \geq 1\} = \lim_{n\to\infty} x_n''\end{aligned}$$

by Lemma 28. □

Lemma 30. *Let $X = \Delta([a, b])$. Let $\{x_n\}$ be a convergent sequence in X, with $x_n \to x^*$. Then there is an increasing sequence $\{x_n'\}$ and an a decreasing sequence $\{x_n''\}$ such that $x_n' \leq x_n \leq x_n''$, and $\lim_{n\to\infty} x_n' = x^* = \lim_{n\to\infty} x_n''$.*

Proof. The set X ordered by first order stochastic dominance is a complete lattice (see, for example, Lemma 3.1 in Kertz and Rösler (2000)). Suppose that $x_n \to x^*$. Define x_n' and x_n'' by $x_n' = \inf\{x_m : n \leq m\}$ and $x_n'' = \sup\{x_m : n \leq m\}$. Clearly, $\{x_n'\}$ is an increasing sequence, $\{x_n''\}$ is decreasing, and $x_n' \leq x_n \leq x_n''$.

Let F_x denote the cdf associated with x. Note that $F_{x_n''}(r) = \inf\{F_{x_m}(r) : n \leq m\}$ while $F_{x_n'}(r)$ is the right-continuous modification of $\sup\{F_{x_m}(r) : n \leq m\}$. For any point of continuity r of F, $F_{x_m}(r) \to F_{x^*}(r)$, so

$$F_x(r) = \sup\{\inf\{F_{x_m}(r) : n \leq m\} : n \geq 1\}$$

by Lemma 28.

Moreover, $F_{x^*}(r) = \inf\{\sup\{F_{x_m}(r) : n \leq m\} : n \geq 1\}$. Let $\varepsilon > 0$. Then

$$F_{x^*}(r-\varepsilon) \leftarrow \sup\{F_{x_m}(r-\varepsilon) : n \leq m\} \leq F_{x_n'}(r) \leq \sup\{F_{x_m}(r+\varepsilon) : n \leq m\} \to F_{x^*}(r+\varepsilon)$$

Then $F_{x_n'}(r) \to F_{x^*}(r)$, as r is a point of continuity of F_{x^*}. □

The following lemma is immediate.

Lemma 31. *Let $X = \mathbf{R}^n_+$ with the standard vector order $\leq$, and let $B = \mathbf{Q}^n_+$. Then the countable order property is satisfied.*

Our last lemma is a direct implication of Theorem 15.11 of Aliprantis and Border (2006).

Lemma 32. *Let $a, b \in \mathbf{R}$, where $a < b$. Let $X = \Delta([a, b])$, the set of Borel probability distributions on $[a, b]$ endowed with the weak* topology. Let B be the set of probability distributions p with finite support on $\mathbf{Q} \cap [a, b]$, where for all $q \in \mathbf{Q} \cap [a, b]$, $p(q) \in \mathbf{Q}$. Then the countable order property is satisfied.*

Appendix C. Proof of Theorems 8, 25, 26 and 10

In this section, we let $\overline{\mathcal{R}}^{\text{mon}}$ denote the set of complete, continuous, and strictly monotonic binary relations. Members of $\overline{\mathcal{R}}^{\text{mon}}$ need not be transitive. Likewise, $\overline{\mathcal{R}}^{\text{ls}}$ is the set of complete, continuous, and locally strict binary relations.

We record the following facts:

Lemma 33. *Let $\succeq$ be a continuous binary relation. If $x \succ y$ then there are neighborhoods V_x of x and V_y of y such that $x' \succ y'$ for all $x' \in V_x$ and $y' \in V_y$.*

Theorem 34. *Suppose that $\succeq$ and $\succeq'$ are two complete and continuous binary relations. Suppose that $\succeq'$ is locally strict, and let $B \subseteq X$ be dense. If $\succeq |_{B \times B} \subseteq \succeq' |_{B \times B}$, then $\succeq \subseteq \succeq'$.*

Proof. Suppose by means of contradiction that there are x, y such that $x \succeq' y$ but $x \succeq y$ is false. Then $y \succ x$, as $\succeq$ is complete, and $x \sim' y$ as $y \succ x$ implies $y \succeq' x$. Let U be a neighborhood of (y, x) so that for all $(y', x') \in U$, we have $y' \succ x'$. By local strictness, let $(y^*, x^*) \in U$ for which $x^* \succ' y^*$. Then $y^* \succ x^*$ as well. Now let V be a neighborhood of (x^*, y^*) for which for all $(x', y') \in V$, $y' \succ x'$ and $x' \succ y'$. Choose $(\hat{x}, \hat{y}) \in V \cap (B \times B)$ and observe that we have a contradiction. □

Theorem 35. *Suppose that $\succeq$ and $\succeq^*$ are complete and continuous binary relations. Suppose that $\succeq^*$ is locally strict. If $\succeq \subseteq \succeq^*$, then $\succeq = \succeq^*$.*

Proof. Suppose by means of contradiction that there is (x, y) such that $x \succeq^* y$ but $x \succeq y$ is false. Conclude that $y \succeq x$, so that $y \succeq^* x$ (and hence $x \sim^* y$) and $y \succ x$. Let U be a neighborhood about (x, y) for which for all $(x', y') \in U$, $y' \succ x'$ (Lemma 33). By local strictness, there is $(x^*, y^*) \in U$ such that $x^* \succ^* y^*$. Hence we have $x^* \succ^* y^*$ and $y^* \succ x^*$, contradicting $\succeq \subseteq \succeq^*$. □

We now prove Theorems 25 and 26.

Proof of Theorem 25. The proof follows directly from Theorems 34 and 35. □

Proof of Theorem 26. First, it is straightforward to show that $x \succ y$ implies $x \succeq' y$. Because otherwise there are x, y for which $x \succ y$ and $y \succ' x$. Take an open neighborhood U about (x, y) and a pair $(z, w) \in U \cap (B \times B)$ for which $z \succ w$ and $w \succ' z$, a contradiction. Symmetrically, we also have $x \succ' y$ implies $x \succeq y$.

Now, without loss, suppose that there is a pair x, y for which $x \succ y$ and $x \sim' y$. By connectedness and continuity, $V = \{z : x \succ z \succ y\}$ is nonempty and by continuity it is open.[10] We claim that there is a pair $(w, z) \in (V \times V) \cap (B \times B)$ for which $w \succ z$. For otherwise, for all $(w, z) \in V \times V \cap (B \times B)$, $w \sim z$. Conclude then by continuity that for all $(w, z) \in V \times V$, $w \sim z$. Observe that this implies that, for any $w \in V$, the set $\{z : w \succ z \succ y\} = \varnothing$, as if $w \succ z \succ y$, we also have that $x \succeq w \succ z$, from which we conclude $x \succ z$, so that $z \in V$ and hence $z \sim w$, a contradiction. Observe that $\{z : w \succ z \succ y\} = \varnothing$ contradicts the continuity of $\succeq$ and the connectedness of X (same argument as nonemptyness of V; see the footnote).

We have shown that there is $(w, z) \in (V \times V) \cap (B \times B)$ for which $w \succ z$, so that $x \succ w \succ z \succ y$. Further, we have hypothesized that $x \sim' y$. By the first paragraph, we know that $x \succeq' w \succeq' z \succeq' y$. If, by means of contradiction, we have $w \succ' z$, then $x \succ' y$, a contradiction. So $w \sim' z$ and $w \succ z$, a contradiction to $\succeq_{B\times B} = \succeq'_{B\times B}$. □

Lemma 36. *Let $A \subseteq X \times X$. Then $\{\succeq : A \subseteq \succeq\}$ is closed in the closed convergence topology.*

Proof. Let $\succeq_n$ be a convergent sequence in the set in question, where $\succeq_n \to \succeq$. Then for all $(x, y) \in A$, we have $x \succeq_n y$, hence $x \succeq y$. So $(x, y) \in \succeq$. □

Lemma 37. *Suppose X is locally compact Polish, and that $<$ has open intervals. Then $\overline{\mathcal{R}}^{mon}$ is closed in the topology of closed convergence.*

[10] The argument for nonemptiness is as follows. If, by means of contradiction, $V = \varnothing$, then $\{z : x \succ z\}$ and $\{z : z \succ y\}$ are nonempty open sets. Further, for any $z \in X$, either $x \succ z$ or $z \succ y$ (because if $\neg(x \succ z)$ then by completeness $z \succeq x$, which implies that $z \succ y$). Conclude that $\{z : x \succ z\} \cup \{z : z \succ y\} = X$ and each of the sets are nonempty and open (by continuity); these sets are disjoint, violating connectedness of X.

Proof. By Lemma 6, since X is locally compact Polish, the topology of closed convergence is compact metrizable.

Suppose $\succeq_n \to \succeq$ where each $\succeq_n$ is continuous, strictly monotonic, and complete. We know that $\succeq$ is continuous by compactness. Suppose by means of contradiction that $\succeq$ is not strictly monotonic, so that there are $x, y \in X$ for which $x > y$ and $y \succeq x$. Then there are $(x_n, y_n) \to (x, y)$ for which $y_n \succeq_n x_n$. For n large, $x_n > y_n$, a contradiction to the fact that $\succeq_n$ is strictly monotonic. Finally, completeness follows as for each x, y, either $x \succeq_n y$ or $y \succeq_n x$, so there is a subsequence n_k for which either $x \succeq_{n_k} y$ or for which $y \succeq_{n_k} x$. □

Lemma 38. *Suppose that B is dense, $\succeq'$ is complete, and each of $\succeq$ and $\succeq^*$ are continuous and locally strict complete relations. Then if*

$$\succeq' |_{B\times B} \subseteq \succeq^* |_{B\times B} \cap \succeq |_{B\times B},$$

it follows that $\succeq^* = \succeq$.

Proof. Suppose, by means of contradiction and without loss of generality, that there are $x, y \in X$ for which $x \succeq^* y$ and $y \succ x$. By continuity of $\succeq$ and local strictness of $\succeq^*$, we can without loss of generality assume that $x \succ^* y$ and $y \succ x$. By continuity of each of $\succeq$ and $\succeq^*$, there exists $a, b \in B$ such that $a \succ^* b$ and $b \succ a$. But by completeness of $\succeq'$, either $a \succeq' b$, contradicting $\succeq' |_{B\times B} \subseteq \succeq |_{B\times B}$, or $b \succeq' a$, contradicting $\succeq' |_{B\times B} \subseteq \succeq^* |_{B\times B}$. □

We now turn to the main proof of the theorem.

Proof of Theorem 8. By Lemma 37, $\overline{\mathcal{R}}^{\text{mon}}$ is compact. Let $\succeq'$ be any strictly monotonic and complete binary relation such that for all k and all $\{x, y\} \in \Sigma_k$, $x \in c^k(\{x, y\})$ if and only if $x \succeq' y$ ($\succeq'$ exists by the projection requirement on choice sequences, and by the fact that $c \sqsubseteq c_{\succeq^*}$).

For each k, let $\succeq'_k = \{(x, y) : \{x, y\} \in \{B_1, \ldots, B_k\}$ and $x \succeq' y\}$.

For each k, let

$$P_k = \{\succeq \in \overline{\mathcal{R}}^{\text{mon}} : \succeq'_k \subseteq \succeq\},$$

the set of relations which weakly rationalize c. Observe by definition that by Lemma 36, P_k is closed, and hence compact. By assumption, each $\succeq \in P_k$ satisfies $\succeq \in \overline{\mathcal{R}}^{\text{ls}}$, and obviously, for all k, $\succeq^* \in P_k$. Further, observe that

$\bigcap_k P_k = \{\succeq^*\}$, since if $\succeq \in \bigcap_k P_k$, by definition $\succeq'_{B\times B} \subseteq \succeq^* |_{B\times B} \cap \succeq |_{B\times B}$ and Lemma 38.

The result now follows as each P_i is compact and $\bigcap_k P_k = \{\succeq^*\}$. That is, let $\succeq_k \in P_k$, which is a decreasing, nested collection of compact sets. Suppose by means of contradiction and without loss that $\succeq_k \rightarrow \succeq' \neq \succeq^*$, and observe then that it follows that $\succeq' \in P_k$ for all k, contradicting $\bigcap_i P_i = \{\succeq^*\}$. □

Proof of Theorem 10. Observe that for any k, the set

$$P_k = \{\succeq \in \overline{\mathcal{R}}^{\text{mon}} : \succeq \text{ weakly rationalizes } c^k\}$$

is closed, and hence compact by Lemma 36. Observe that $\mathcal{P}^k(c) \subseteq P_k$. Moreover, it is obvious that $P_{k+1} \subseteq P_k$. Suppose that there is no k for which $\mathcal{P}^k(c) = \varnothing$. Then, since each $P_k \neq \varnothing$ and each P_k is compact, $\bigcap_k P_k \neq \varnothing$. Let $\succeq^* \in \bigcap_k P_k$.

We claim that $\bigcap_k P_k = \{\succeq^*\}$. Suppose by means of contradiction that there is $\succeq \neq \succeq^*$ where $\succeq \in \bigcap_k P_k$. Let $\succeq'$ be any complete relation such that for all $(a,b) \in B \times B$, $a \succeq' b$ if and only if $a \in c^k(\{a,b\})$, for k such that $\{a,b\} \in \Sigma^k$. Then, by definition of weak rationalization, we have $\succeq'_{B\times B} \subseteq \succeq_{B\times B} \cap \succeq^*_{B\times B}$. Appeal to Lemma 38 to conclude that $\succeq = \succeq^*$, a contradiction.

Finally, since $\bigcap_k P_k = \{\succeq^*\}$, and each P_k is compact, it follows that $\lim_{k\to\infty} \operatorname{diam}(P_k) \rightarrow 0$.[11] Hence, since $0 \leq \operatorname{diam}(\mathcal{P}^k(c)) \leq \operatorname{diam}(P_k)$, the result follows.

□

Appendix D. Proof of Theorem 12

The set of weakly monotone and continuous binary relations is compact in the topology of closed convergence. Suppose wlog that $\succeq^k \rightarrow \succeq$. Then $\succeq$ is a continuous binary relation. We shall prove that $\succeq = \succeq^*$.

First we show that $x \succ^* y$ implies that $x \succ y$. So let $x \succ^* y$. Let U and V be neighborhoods of x and y, respectively, such that $x' \succ^* y'$ for all $x' \in U$ and $y' \in V$. Such neighborhoods exist by the continuity of $\succeq^*$. We prove first that if $(x', y') \in U \times V$, then there exists N such that $x' \succ_n y'$ for all $n \geq N$. By hypothesis, there exist $x'' \in U \cap B$ and $y'' \in U \cap B$ such that $x'' \leq x'$ and

[11] Otherwise, we could choose $\epsilon > 0$ and two subsequences $\succeq_{k_l}, \succeq'_{k_l}$ such that $\delta_C(\succeq_{k_l}, \succeq'_{k_l}) \geq \epsilon$ and $\succeq_{k_l} \rightarrow \succeq \in \bigcap_k P_k$ and $\succeq'_{k_l} \rightarrow \succeq' \in \bigcap_k P_k$ where $\delta_C(\succeq, \succeq') \geq \epsilon$, a contradiction.

$y' \leq y''$. Each $\succeq_n$ is a strong rationalization of the finite experiment of order n, so if $\{\tilde{x}, \tilde{y}\} \in \Sigma_n$ then $\tilde{x} \succ_n \tilde{y}$ implies that $\tilde{x} \succ_m \tilde{y}$ for all $m \geq n$. Since $x'', y'' \in B$, there is N is such that $\{x'', y''\} \in \Sigma_N$. Thus $x'' \succ^* y''$ implies that $x'' \succ_n y''$ for all $n \geq N$. So, for $n \geq N$, $x' \succ_n y'$, as $\succeq_n$ is weakly weakly monotone.

Now we establish that $x \succ y$. Let $\{(x_n, y_n)\}$ be an arbitrary sequence with $(x_n, y_n) \to (x, y)$. By hypothesis, there is an increasing sequence $\{x'_n\}$, and a decreasing sequence $\{y'_n\}$, such that $x'_n \leq x_n$ and $y_n \leq y'_n$ while $(x, y) = \lim_{n\to\infty}(x'_n, y'_n)$.

Let N be large enough that $x'_N \in U$ and $y'_N \in V$. Let $N' \geq N$ be such that $x'_N \succ_n y'_N$ for all $n \geq N'$ (we established the existence of such N' above). Then, for any $n \geq N'$ we have that

$$x_n \geq x'_n \geq x'_N \succ_n y'_N \geq y'_n \geq y_n.$$

By the weak monotonicity of $\succeq_n$, then, $x_n \succ_n y_n$. The sequence $\{(x_n, y_n)\}$ was arbitrary, so $(y, x) \notin \succeq = \lim_{n\to\infty} \succeq_n$. Thus $\neg(y \succeq x)$. Completeness of $\succeq$ implies that $x \succ y$.

In second place we show that if $x \succeq^* y$ then $x \succeq y$, thus completing the proof. So let $x \succeq^* y$. For any $k \geq 1$, choose $x' \in N_x(1/k) \cap B$ with $x' \geq x$, and $y' \in N_y(1/k) \cap B$ with $y' \leq y$; so that $x' \succeq^* x \succeq^* y \succeq^* y'$, as $\succeq^*$ is weakly weakly monotone. Recall that $\succeq_n \uparrow$. So $x' \succeq^* y'$ and $x', y' \in B$ imply that $x' \succeq_n y'$ for all n large enough. Let $n_k \geq n_{k-1}$ such that $x' \succeq_{n_k} y'$; and let $x' = x_{n_k}$ and $y' = y_{n_k}$.

Then we have $(x^{n_k}, y^{n_k}) \to (x, y)$ and $x_{n_k} \succeq_{n_k} y_{n_k}$. Thus $x \succeq y$.

Appendix E. Proof of Theorem 19

We begin with two useful lemmas.

Lemma 39. *Φ is an open map if for any $u^* \in \mathcal{U}$ and any sequence $\succeq_n$ in $\mathcal{R}$ with $\succeq_n \to \Phi(u^*)$, there is a sequence $\{u_n\}$ in $\mathcal{U}$ such that $u_n \in \Phi^{-1}(\succeq_n)$ and $u_n \to u^*$ in the topology of compact convergence.*

Proof. Suppose that there is $V \subseteq \mathcal{U}$ open, but $\Phi(V)$ is not open. Then there is $u^* \in V$ and $\succeq_n \notin \Phi(V)$ such that $\succeq_n \to \Phi(u^*)$ (since closed convergence topology is metrizable). Since $u^* \in V$, any sequence $u_n \in \Phi^{-1}(\succeq_n)$ for which

$u_n \to u^*$ eventually has $u_n \in V$. But if u_n is chosen to represent $\succeq_n$, this implies that $\Phi(u_n) \in \Phi(V)$ for n large, a contradiction. □

Lemma 40. *For any $\succeq$ and $x \in X$, there is a unique $m^*(x) \in M$ with $x \sim m^*(x)$. Moreover, if we fix $u \in \mathcal{U}$ then the function $u_{\succeq} : X \to \mathbf{R}$ defined by $u_{\succeq}(x) = u(m^*(x))$ is a continuous utility representation of $\succeq$.*

Proof. Consider the sets $A = \{m \in M : m \succeq x\}$ and $B = \{m \in M : x \succeq m\}$. These sets are closed because $\succeq$ is continuous, their union is M as $\succeq$ is complete, and they are nonempty as $\succeq$ is monotone and there exist $\underline{m}, \overline{m} \in M$ with $\underline{m} \leq x \leq \overline{m}$ by our hypothesis on M. M is connected, so A and B cannot be disjoint; hence there is $m \in M$ with $x \sim m$. This m must be unique because M is totally ordered, and $\succeq$ is strictly monotone.

We now show that $u_{\succeq}$ is a continuous utility representation of $\succeq$. Let $x \succeq y$. Then transitivity and monotonicity of $\succeq$ imply that $m^*(x) \geq m^*(y)$. Thus $u_{\succeq}(x) = u^*(m^*(x)) \geq u^*(m^*(y)) = u_{\succeq}(y)$. The converse implications hold as well; thus $u_{\succeq}$ represents $\succeq$.

To prove continuity, let $x_n \to x^*$. We shall prove that $m_n = m^*(x_n) \to m^*(x^*) = \hat{m}$. Suppose first that $\hat{m}$ is not the largest or the least element of M. For each neighborhood U of $\hat{m}$ there exists, by our hypothesis on M, $\underline{m}, \overline{m} \in M$ with $\underline{m} < \hat{m} < \overline{m}$ and $[\underline{m}, \overline{m}] \subseteq U$. Then

$$V = \{z \in X : \overline{m} \succ z\} \cap \{z \in X : z \succ \underline{m}\}$$

is a neighborhood of x^*, as $x^* \sim \hat{m}$ and $\succeq$ is continuous and monotone. For large enough n then $x^n \in V$, so $m^n \in [\underline{m}, \overline{m}] \subseteq U$. Suppose now that $\hat{m}$ is the largest element of M. Then, reasoning as above, $x^n \in \{z \in X : z \succ \underline{m}\}$ for all large enough n, so that $\underline{m} \leq m^n$. We have $m^n \leq \overline{m}$ as $\overline{m}$ is the largest element of M. Thus $m^n \in [\underline{m}, \overline{m}] \subseteq U$. The argument when $\underline{m}$ is the least element of M is analogous. □

We now turn to the main proof of the theorem.

Proof of Theorem 19. Let $u^* \in \mathcal{U}$ and $\{\succeq_n\}$ be a sequence in $\mathcal{R}$ with $\succeq_n \to \Phi(u^*)$. By Lemma 39 it is enough to exhibit a sequence $u_n \in \Phi^{-1}(\succeq_n)$ and $u_n \to u^*$ in the topology of compact convergence.

Let $u_n = u_{\succeq_n}$ as defined in Lemma 40 from u^*. Lemma 40 implies that $u_n \in \Phi^{-1}(\succeq_n)$. By XII Theorem 7.5 p. 268 of Dugundji (1966), to establish compact convergence it is enough to show that for any convergent sequence $\{x_n\}$, with $x_n \to x^*$, $u_n(x_n) \to u^*(x^*)$.

To this end, let $x_n \to x^*$. Let $\hat{m} = m^*(x^*)$ and $m_n \sim_n x_n$, using the notation in Lemma 40, and U be a neighborhood of $\hat{m}$. Let $\underline{m}, \overline{m} \in M$ be such that $\underline{m} < \hat{m} < \overline{m}$ and $[\underline{m}, \overline{m}] \subseteq U$. Then it must be true that $m^n \in [\underline{m}, \overline{m}]$ for all n large enough. To see this, note that if, for example, $m^n \geq \overline{m}$ infinitely often then there would exist a subsequence for which $x^n \succeq^n m^n \succeq \overline{m}$ (by monotonicity of $\succeq$), which would imply that $x^* \succeq \overline{m} > \hat{m}$, as $\succeq_n \to \succeq$. But $\hat{m} \sim x^* \succeq \overline{m}$ is a violation of monotonicity.

Now $m^n \in [\underline{m}, \overline{m}] \subseteq U$ for all n large enough means that $m^n \to \hat{m}$. Thus

$$u_{\succeq_n}(x_n) = u^*(x_n) \to u^*(x^*) = u_{\succeq}(x^*),$$

as u^* is continuous. □

Appendix F. Proof of Theorem 22

By Theorem 8 of Border and Segal (1994), $\Phi(\mathcal{V})$ is compact, and therefore $\succeq_k$ possesses a limit point $\succeq^* \in \Phi(\mathcal{V})$. By Lemma 36, the set of $\succeq_k$ weakly rationalizing c^k is closed, and hence compact. Suppose by means of contradiction that there is some $\succeq'_k$ also weakly rationalizing c^k which converges to $\succeq \neq \succeq^*$. Observe that each of $\succeq^*$ and $\succeq$ weakly rationalize each c^k.

Finally, let $\succeq'$ be any complete relation such that for all $(a, b) \in B \times B$, $a \succeq' b$ if and only if $a \in c^k(\{a, b\})$, for k such that $\{a, b\} \in \Sigma^k$. Then, by definition of weak rationalization, we have $\succeq'_{B\times B} \subseteq \succeq_{B\times B} \cap \succeq^*_{B\times B}$. Appeal then to Lemma 38 to conclude that $\succeq = \succeq^*$, a contradiction.